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On Analytic Modeling of Lunar Perturbations of Artificial Satellites of the Earth

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FOR THE COMMANDER

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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Group 91

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ABSTRACT

Two different procedures for analytically modeling the effects of the Moon's direct gravitational force on artificial Earth satellites are discussed from theoretical and numerical viewpoints. One is developed using classical series expansions of inclination and eccentricity for both the satellite and the Moon, and the other employs the method of averaging.

Both solutions are seen to have advantages, but it is shown that while the former is more accurate in special situations, the latter is quicker and more practical for the general orbit determination problem where observed data are used to correct the orbit in near real time.

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1. INTRODUCTION

Analytical theories describing the perturbations of artificial satellites influenced by the direct gravitational force of the Moon typically involve many simplifying assumptions, for which there is little comment in the literature as to their effect upon the overall accuracy of the resulting theory. For instance, the potential which gives rise to the disturbing function is fundamentally of two degrees of freedom involving a complicated coupling of the motion of the Moon with that of the satellite. Since the disturbing function contains an exact generating function for Legendre polynomials, the problem is simplified somewhat by expanding in terms of these polynomials and truncating to include just the second, or sometimes also the third, harmonic (where the first is absent). In order to simplify the problem further, the motion of the Moon is assumed to be a pure Kepler mean elliptical orbit with constant linear rates for the angular quantities. Of course, this differs substantially from the true motion for the Moon. Finally, in order to perform a completely analytical integration of the equations of motion for the satellite, further simplifying assumptions are imposed, such as ignoring some of the less dominant coordinates over the integration or truncating the disturbing function to simplify the solution.

In this work, two philosophically different solutions will be presented and compared in both a theoretical and numerical framework. It is the intent for the numerical examples to highlight the simplifying issues mentioned above. The first method to be presented will be called the "series expansion method" and follows the development of Kaula [1] and Giacaglia [2]. We will refer to this solution as the (SEM) solution. The second solution is developed using a method of averaging and follows the work of Kozai [3], Sridharan and Seniw [4], and Hujsak [5]. We will refer to this solution as the (MAV) solution. The third harmonic is included in the (MAV) solution discussed in this report (which is given in its entirety in the appendix) since it tends to be significant in accurately modeling lunar perturbations of high-altitude satellite orbits.

The gravitational force from a third body acting on a point source can be described (assuming a reference system defined by the center of the Earth) using Newton's law of gravitation by the potential

$$R_k = \frac{Gm_k}{\rho_k} - \frac{Gm_k(\mathbf{r}_k \cdot \mathbf{r})}{r_k^3}, \tag{1.1}$$

where $\rho_k = |\,\mathbf{r} - \mathbf{r}_k\,|$ is the magnitude of the vector difference of the vector $\mathbf{r}$ pointing to the object and the vector $\mathbf{r}_k$ pointing to the third body, Gm_k is Newton's gravitational constant times the mass of the third body, and r_k denotes the magnitude of $\mathbf{r}_k$. For the Moon, the size of Gm_k can be approximated by

$$Gm_k = N_k^2 a_k^3,$$

where $N_k^2 \simeq 1.59 \times 10^{-5}\text{rev}^2/\text{day}^2$ and a_k is the semimajor axis of the lunar orbit about the Earth. Using the generating function for Legendre polynomials in $1/\rho_k$, we arrive at the disturbing function for the Moon, which can be written as

$$R_k = \frac{Gm_k}{r_k} \sum_{l \geq 2} (\frac{r}{r_k})^l P_l(\cos \psi_k), \tag{1.2}$$

where r is the range to the object and ψ_k is the geocentric elongation of the satellite from the Moon (i.e., $rr_k \cos \psi_k = \mathbf{r} \cdot \mathbf{r}_k$). It is worth noting that the effects of the Moon are as large or larger than the flatness of the Earth perturbation for high-altitude orbits, but are much smaller for low-altitude satellites.

Kozai [3], Sridharan and Seniw [4], and Hujsak [5] all truncated the disturbing force to $l = 2$, but the development of Kaula [1] and Giacaglia [2] is shown for all $l \geq 2$. Also Musen, Bailie, and Upton [6] included the third harmonic in their analysis; however, the ephemeris of the Moon in their study relies on a Kepler element set referred to the equator of the Earth. This was also done in Kaula's treatment [1]. It was noted by Kozai [3,7] that it is more desirable to develop a theory which relies on a Kepler ephemeris for the Moon referred to the ecliptic plane since then the inclination is roughly constant and the longitude of the right ascending node can be approximated by a simple linear function of time. The work presented in this report assumes this reference system for the lunar ephemeris while the satellite elements are referred to the equator. This will involve some kind of geometrical rotation in both theories.

2. THE SERIES EXPANSION SOLUTION

The (SEM) solution has the primary advantage of allowing a simultaneous integration of the two fundamental frequencies involved in the problem, the mean motion of the satellite and the mean motion of the Moon. As an added benefit of the following analysis, the secular rates of the argument of perigee and right ascending node of the satellite and the Moon can also be included in the integration at the same time. The idea is classical in principle and transforms the disturbing function so that the faster angular variables are converted to explicit functions of time on the right-hand side of the system of differential equations so that a direct solution is possible. The development of the solution is complicated and, for the most part, quite handily displayed in [2]. However, there are some algebraic errors at crucial junctures there, and so a complete development will be presented here with some of the more evident mathematical details omitted.

We can express (1.2) using Legendre's addition theorem

$$R_k = \sum_{l\geq 2}\sum_{m=0}^{l} \frac{Gm_k\epsilon_m(l-m)!}{a_k(l+m)!}(\frac{a}{a_k})^l(\frac{r}{a})^l(\frac{a_k}{r_k})^{l+1}$$
$$P_l^m(\sin\delta')P_l^m(\sin\delta)\cos m(\alpha-\alpha'), \tag{2.1}$$

where $\alpha, \delta, \alpha', \delta'$ are the right ascension and declination, referred to the equator, of the satellite and the Moon, respectively:

$$\epsilon_m = \begin{cases} 1 & \text{if } m = 0 \\ 2 & \text{if } m \neq 0 \end{cases};$$

a, a_k denote the semimajor axes of the satellite and the Moon, respectively; and $P_l^m(x)$ denotes the associated Legendre function of degree l and order m. If we temporarily define

$$A_l^m \equiv \frac{Gm_k\epsilon_m(l-m)!}{a_k(l+m)!}(\frac{a}{a_k})^l(\frac{r}{a})^l(\frac{a_k}{r_k})^{l+1}P_l^m(\sin\delta')$$

and let $C_l^m \equiv A_l^m \cos m\alpha'$ and $S_l^m \equiv A_l^m \sin m\alpha'$, then we can write

$$R_k = \sum_{l\geq 2}\sum_{m=0}^{l} P_l^m(\sin\delta)[C_l^m\cos m\alpha + S_l^m \sin m\alpha].$$

One can then apply Kaula's inclination functions [1] and express

$$R_k = \sum_{l\geq 2}\sum_{m=0}^{l}\sum_{p=0}^{l} F_{lmp}(I)$$
$$\left\{\begin{bmatrix} C_l^m \\ -S_l^m \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \cos\theta_{lmp} + \begin{bmatrix} S_l^m \\ C_l^m \end{bmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} \sin\theta_{lmp}\right\},$$

where $\theta_{lmp} = (l-2p)(\omega + f) + m\Omega$ and ω, f, and Ω denote the argument of perigee, true anomaly, and right ascending node of the satellite, respectively, and $F_{lmp}(I)$ denotes a function of the satellite inclination I.

In case that Kepler elements are chosen to represent the Moon's ephemeris and that they are referred to the ecliptic plane (rather than the equator as in the satellite case), a rotation of the spherical harmonics for the Moon is needed [2, Section 3]. We can write

$$P_l^m(\sin\delta')\exp[im\alpha'] = \sum_{s=-l}^{l} \Lambda_l^{m,s} P_l^s(\sin\delta_k)\exp[is\alpha_k],$$

where $\Lambda_l^{m,s}$ is a function of the ecliptic inclination ι, and δ_k, α_k denote the declination and right ascension of the Moon referred to the ecliptic plane, respectively. The expression for $\Lambda_l^{m,s}$ below is taken directly from [8],

$$\Lambda_l^{m,s}(\iota) = \frac{(l-s)!}{(l-m)!}\exp[i(m-s)\frac{\pi}{2}]U_l^{m,s}(\iota),$$

where

$$\begin{aligned} U_l^{m,s}(\iota) &= (-1)^{m-s} \sum_{r=\max[0,-(m+s)]}^{\min[l-s,l-m]} (-1)^{l-m-r} \binom{l+m}{m+s+r} \\ &\quad \binom{l-m}{r} \cos^{m+s+2r}(\frac{\iota}{2})\sin^{-m-s+2(l-r)}(\frac{\iota}{2}). \end{aligned}$$

Noting that

$$P_l^{-s}(x) = (-1)^s\frac{(l-s)!}{(l+s)!}P_l^s(x),$$

we can define

$$C_l^{m,s} \equiv \frac{1}{2}[U_l^{m,s} + (-1)^s U_l^{m,-s}],$$

$$S_l^{m,s} \equiv \frac{1}{2}[U_l^{m,s} - (-1)^s U_l^{m,-s}],$$

and

$$A_l^{m,s} \equiv \frac{Gm_k\epsilon_m\epsilon_s(l-s)!}{a_k(l+m)!}(\frac{a}{a_k})^l(\frac{r}{a})^l(\frac{a_k}{r_k})^{l+1},$$

so that

$$C_l^m + iS_l^m = (i)^m \sum_{s=0}^{l} A_l^{m,s} P_l^s(\sin\delta_k)$$
$$\{C_l^{m,s} \cos[s(\alpha_k - \frac{\pi}{2})] + iS_l^{m,s} \sin[s(\alpha_k - \frac{\pi}{2})]\}.$$

Since if m is even $(i)^m = (-1)^{\frac{m}{2}}$ and if m is odd $(i)^m = i(-1)^{\frac{(m-1)}{2}}$, it follows that

$$(-1)^{k_1} \sum_{s=0}^{l} A_l^{m,s} P_l^s(\sin\delta_k) C_l^{m,s} \cos[s(\alpha_k - \frac{\pi}{2})] = \begin{cases} C_l^m & \text{if } m \text{ is even} \\ S_l^m & \text{if } m \text{ is odd} \end{cases};$$

and

$$(-1)^{k_1} \sum_{s=0}^{l} A_l^{m,s} P_l^s(\sin\delta_k) S_l^{m,s} \sin[s(\alpha_k - \frac{\pi}{2})] = \begin{cases} S_l^m & \text{if } m \text{ is even} \\ -C_l^m & \text{if } m \text{ is odd} \end{cases}.$$

where $k_1 = \|\frac{m}{2}\|$, the greatest integer part of $\frac{m}{2}$. Expanding further in terms of Kaula's inclination functions once again (this time in terms of the inclination I_k of the Moon referred to the ecliptic plane), we can write

$$(-1)^{k_1} \sum_{s=0}^{l} \sum_{q=0}^{l} A_l^{m,s} F_{lsq}(I_k) \left\{ \begin{bmatrix} C_l^{m,s} \\ 0 \end{bmatrix}_{l-s \text{ odd}}^{l-s \text{ even}} \cos\theta'_{lsq} \right.$$
$$\left. + \begin{bmatrix} 0 \\ C_l^{m,s} \end{bmatrix}_{l-s \text{ odd}}^{l-s \text{ even}} \sin\theta'_{lsq} \right\} = \begin{cases} C_l^m & \text{if } m \text{ is even} \\ S_l^m & \text{if } m \text{ is odd} \end{cases}, \tag{2.2}$$

and

$$(-1)^{k_1} \sum_{s=0}^{l} \sum_{q=0}^{l} A_l^{m,s} F_{lsq}(I_k) \left\{ \begin{bmatrix} 0 \\ -S_l^{m,s} \end{bmatrix}_{l-s \text{ odd}}^{l-s \text{ even}} \cos\theta'_{lsq} \right.$$
$$\left. + \begin{bmatrix} S_l^{m,s} \\ 0 \end{bmatrix}_{l-s \text{ odd}}^{l-s \text{ even}} \sin\theta'_{lsq} \right\} = \begin{cases} S_l^m & \text{if } m \text{ is even} \\ -C_l^m & \text{if } m \text{ is odd} \end{cases}, \tag{2.3}$$

where $\theta'_{lsq} = (l-2q)(\omega_k + f_k) + s(\Omega_k - \frac{\pi}{2})$ and ω_k, f_k, and Ω_k denote the argument of perigee, the true anomaly, and the right ascending node of the lunar orbit referred to the ecliptic plane, respectively.

Since algebraic errors appear in the mathematics leading to equation (15) in [2], equations (15 to 19) there are false; but the form of (19) is desirable, and so the mathematics which follow will be aimed toward the objective of producing a similar form. Suppose m is even and $l - m$ is even, then

$$R_k = \sum_{l\geq 2} \sum_{m=0}^{l} \sum_{p=0}^{l} F_{lmp}(I) \left(C_l^m \cos\theta_{lmp} + S_l^m \sin\theta_{lmp} \right),$$

and with some tedious algebra, we arrive at

$$R_k = \sum_{l\geq 2} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{s=0}^{l} \sum_{q=0}^{l} (-1)^{k_1} A_l^{m,s} F_{lmp}(I) F_{lsq}(I_k) \Theta_{lmpsq}, \tag{2.4}$$

where Θ_{lmpsq} can be expressed as

$$\Theta_{lmpsq} = \frac{1}{2} \begin{bmatrix} (-1)^s U_l^{m,-s} \cos(\theta_{lmp} + \theta'_{lsq}) + U_l^{m,s} \cos(\theta_{lmp} - \theta'_{lsq}) \\ (-1)^s U_l^{m,-s} \sin(\theta_{lmp} + \theta'_{lsq}) - U_l^{m,s} \sin(\theta_{lmp} - \theta'_{lsq}) \end{bmatrix}_{l-s \text{ odd}}^{l-s \text{ even}}.$$

If m is even and $l - m$ is odd, then (2.4) is true with Θ_{lmpsq} expressed as

$$\Theta_{lmpsq} = \frac{1}{2} \begin{bmatrix} (-1)^s U_l^{m,-s} \sin(\theta_{lmp} + \theta'_{lsq}) + U_l^{m,s} \sin(\theta_{lmp} - \theta'_{lsq}) \\ -(-1)^s U_l^{m,-s} \cos(\theta_{lmp} + \theta'_{lsq}) + U_l^{m,s} \cos(\theta_{lmp} - \theta'_{lsq}) \end{bmatrix}_{l-s \text{ odd}}^{l-s \text{ even}}.$$

If m is odd and $l - m$ is even, then (2.4) is true with Θ_{lmpsq} expressed as

$$\Theta_{lmpsq} = \frac{1}{2} \begin{bmatrix} (-1)^s U_l^{m,-s} \sin(\theta_{lmp} + \theta'_{lsq}) + U_l^{m,s} \sin(\theta_{lmp} - \theta'_{lsq}) \\ -(-1)^s U_l^{m,-s} \cos(\theta_{lmp} + \theta'_{lsq}) + U_l^{m,s} \cos(\theta_{lmp} - \theta'_{lsq}) \end{bmatrix}_{l-s \text{ odd}}^{l-s \text{ even}}.$$

If m is odd and $l - m$ is odd, then (2.4) is true with Θ_{lmpsq} expressed as

$$\Theta_{lmpsq} = \frac{1}{2} \begin{bmatrix} -(-1)^s U_l^{m,-s} \cos(\theta_{lmp} + \theta'_{lsq}) - U_l^{m,s} \cos(\theta_{lmp} - \theta'_{lsq}) \\ -(-1)^s U_l^{m,-s} \sin(\theta_{lmp} + \theta'_{lsq}) + U_l^{m,s} \sin(\theta_{lmp} - \theta'_{lsq}) \end{bmatrix}_{l-s \text{ odd}}^{l-s \text{ even}}.$$

Employing the identity $\cos(x - \frac{\pi}{2}) = \sin x$, we can eliminate the need for the case decisions on the indexing parameters by using the following scheme. Let $y_s = \frac{s}{2} - \|\frac{s}{2}\|$ so that $y_s = 0$ if s is even and $y_s = \frac{1}{2}$ if s is odd. By subtracting $y_s\pi$ from all of the angular arguments of the sines and cosines in the above expressions, we can write

$$\begin{aligned} \Theta_{lmpsq} &= \frac{1}{2} \Big\{ (-1)^{k_2} U_l^{m,-s} \cos(\theta_{lmp} + \theta'_{lsq} - y_s\pi) \\ &\quad + (-1)^{k_3} U_l^{m,s} \cos(\theta_{lmp} - \theta'_{lsq} - y_s\pi) \Big\}, \end{aligned} \tag{2.5}$$

where one solution for k_2 and k_3 can be expressed as

$$k_2 = t(m + s - 1) + 1$$
$$k_3 = t(m + s),$$

with $t = (l-1)(\text{mod } 2)$ (that is, $t = 0$ if $l-1$ is even and $t = 1$ if $l-1$ is odd). Substituting the proper definition for $A_l^{m,s}$ and the right side of (2.5) for Θ_{lmpsq}, we can rewrite (2.4) as

$$\begin{aligned} R_k &= \sum_{l\geq 2}\sum_{m=0}^{l}\sum_{s=0}^{l}\sum_{p=0}^{l}\sum_{q=0}^{l}(-1)^{k_1}\frac{Gm_k\epsilon_m\epsilon_s(l-s)!}{2a_k(l+m)!}(\frac{a}{a_k})^l(\frac{r}{a})^l(\frac{a_k}{r_k})^{l+1} \\ & F_{lmp}(I)F_{lsq}(I_k)\Big\{(-1)^{k_2}U_l^{m,-s}\cos(\theta_{lmp}+\theta'_{lsq}-y_s\pi) \\ & +(-1)^{k_3}U_l^{m,s}\cos(\theta_{lmp}-\theta'_{lsq}-y_s\pi)\Big\}. \end{aligned} \tag{2.6}$$

Now Giacaglia proceeded to average (2.6) over the mean anomaly of the satellite to produce the secular and long periodic parts of the disturbing function together with the short period corrections, and a solution is certainly possible by this route, but since the real advantage of (2.6) lies in the fact that it paves the way for simultaneous integration of the angular rates for the satellite and the Moon, we will defer averaging techniques to the next section where all averaging is performed on the outset directly from (1.2). Therefore, we will skip to (117) in [2], where the true anomalies of the satellite and the Moon are converted to their respective mean anomalies via the Hansen coefficients, which satisfy the equation

$$(\frac{r}{a})^n\exp(imf) = \sum_{r=-\infty}^{\infty} H_{m+r}^{n,m}(e)\exp[i(m+r)M].$$

The final form of the series expansion of R_k then becomes

$$\begin{aligned} R_k &= \sum_{l\geq 2}\sum_{m=0}^{l}\sum_{s=0}^{l}\sum_{p=0}^{l}\sum_{q=0}^{l}\sum_{j=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}(-1)^{k_1}\frac{Gm_k\epsilon_m\epsilon_s(l-s)!}{2a_k(l+m)!}(\frac{a}{a_k})^l \\ & F_{lmp}(I)F_{lsq}(I_k)H_{l-2q+r}^{-(l+1),l-2q}(e_k)H_{l-2p+j}^{l,l-2p}(e) \\ & \Big\{(-1)^{k_2}U_l^{m,-s}\cos(\overline{\theta}_{lmpj}+\overline{\theta}'_{lsqr}-y_s\pi) \\ & +(-1)^{k_3}U_l^{m,s}\cos(\overline{\theta}_{lmpj}-\overline{\theta}'_{lsqr}-y_s\pi)\Big\}, \end{aligned} \tag{2.7}$$

where $\overline{\theta}_{lmpj} = (l-2p)\omega + (l-2p+j)M + m\Omega$ and $\overline{\theta}'_{lsqr} = (l-2q)\omega_k + (l-2q+r)M_k + s(\Omega_k - \frac{\pi}{2})$.

The Lagrange equations of motion of the six Kepler elements are listed in [9, p. 29] and are reproduced below for the convenience of the reader :

$$\begin{aligned} \dot{a} &= \frac{2}{na}\frac{\partial R}{\partial M} \\ \dot{e} &= \frac{(1-e^2)}{na^2e}\frac{\partial R}{\partial M} - \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial \omega} \end{aligned}$$

$$
\begin{aligned}
\dot{\omega} &= \frac{-\cos I}{na^2\sqrt{1-e^2}\sin I}\frac{\partial R}{\partial I} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} \\
\dot{I} &= \frac{1}{na^2\sqrt{1-e^2}\sin I}\left[\cos I\frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}\right] \\
\dot{\Omega} &= \frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial R}{\partial I} \\
\dot{M} &= n - \frac{(1-e^2)}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a}.
\end{aligned}
$$

If the mean motion n is approximated as $n = n_0(1 - \frac{3}{2}\frac{\delta a}{a})$, where δa is the perturbation of the semimajor axis due to $\dot{a}$ above, and R_k is substituted for R in the Lagrange equations, then a complete integration of the system is possible provided $a, a_k, e, e_k, I, I_k, \iota, \dot{M}, \dot{M}_k, \dot{\omega}, \dot{\omega}_k, \dot{\Omega}, \dot{\Omega}_k$ are all held fixed and constant throughout the integration. Numerical tests show that this approximation is improved over the long term if the mean values for these constants are used instead of the initial value conditions. Since this type of solution is explained in detail in [2], the mathematical details of this straightforward integration will not be written out here. All of the work in this section carries over routinely for describing the motion disturbed by the Sun's gravitational force, but the expansion is simpler to express since the rotation of the spherical harmonics is not needed (because the Kepler elements for the solar ephemeris can remain in reference to the equatorial frame).

One final note should be made in this section regarding the (SEM) solution. The expansion involving the Hansen coefficients is valid for all eccentricities less than one, although the convergence of these infinite series is slow for high eccentricity. The interested reader is encouraged to consult [10] for a detailed study of this point. Moreover, it is well known that the sum over j cannot be truncated to just a few values if the eccentricity is high, although the series over r can be truncated to $-3 \le r \le 3$ involving the eccentricity of the Moon ($e_k \simeq .054$). These issues make the (SEM) solution extremely impractical for use with all satellites. Even if a satellite has small eccentricity, a truncation of R_k in (2.7) to $l = 2, 3$ with $-3 \le j \le 3$ can involve literally thousands of terms in any one solution. This has the sobering effect of causing the analytical solution outlined here to be actually *slower* than a full-blown numerical integration over (not too) small time scales, say, less than five days. The argument here is a strong motivation for developing a solution by a method which does not resort to any expansions save the first one involving Legendre polynomials, and which requires the same computation power no matter the satellite parameters. Nevertheless, the (SEM) solution is mathematically pleasing since it displays the entire solution for all harmonics in a single compact form, and the dissection of the perturbations into secular, long periodic, and short periodic parts can be readily made and studied.

3. THE METHOD OF AVERAGING SOLUTION

The (MAV) solution to be described in this report is derived by an entirely different approach than the (SEM) solution. For starters, averaging techniques look for closed form expressions of a solution without resorting to infinite series expansions of the type outlined in the previous section. In order to meet this objective, the differential equations are transformed so that the faster periodic variables are eliminated and the resulting equations become simpler to integrate. Averaging methods are particularly useful in situations where more than one degree of freedom is present, and in the case at hand, it is assumed that the mean anomaly of the satellite and the mean anomaly of the Moon are fundamental periodic quantities to be averaged. The averaging allows integration of the equations of motion by one variable at a time, and hence, series expansions in terms of eccentricity are avoided. It will be seen that the (MAV) solution produces useful results and is appropriate for a wide range of satellite applications because it does not have this inherent difficulty with high eccentricity.

A few words should be mentioned concerning the averaging method used here. In some ways. the technique is similar to averaging methods used in other satellite orbit problems (such as the main satellite problem). but some shortcuts have been taken (when sufficient evidence is warranted) because of the complicated nature of our problem at hand. In the von-Zeipel method, for example. the theory of canonical transformations is exploited and a generating function is used to transform the old Hamiltonian and old coordinates to a new Hamiltonian with new coordinates such that a high frequency variable is eliminated and the differential equations become simpler to solve. Successive transformations lead to constant action variables and to angle variables with constant linear rates. which become the foundation for a complete solution. As long as no resonances exist, this technique has been used for varied problems in mechanics with favorable results. The generating functions are typically constructed by using Taylor series expansions and integrating the appropriate partial derivative components, and truncation allows the expressions to be simplified to desired orders. Generating functions define the transformation between new and old coordinates, and this process of transformation is usually referred to by saying that the old coordinates are obtained by adding periodic corrections to the new variables. An excellent example of this method used in practice is Brouwer's solution of the main satellite problem [13]. The generalized method of averaging relies on asymptotic expansions to accomplish the same goal, only the canonical form is not necessarily preserved after transformation. It is shown in [11] how the von-Zeipel method is a special case of the generalized method of averaging (at least through second order), and it is argued that many techniques which require specific transformations can fall under the more powerful and general category of asymptotic expansions.

In our use of the method of averaging, a solution is desired which is accurate to first order only. and so the disturbing function is averaged directly in the spirit of [3]. The differential equations are separated into their secular (those which have the mean anomalies of the satellite and the Moon absent), long periodic (those which have an explicit dependence on the mean anomaly of the Moon). and short periodic (those which have an explicit dependence on the mean anomaly of the satellite) parts. This is entirely consistent with the generalized method of averaging (to first order):

however, after two transformations by averaging these anomalies, it turns out that the transformed action variables are not yet constant. The rates of the eccentricity and inclination are not zero since the argument of perigee and right ascending node of the satellite and the Moon have not been averaged out. Nevertheless, the integration is still proceeded through in the same spirit of the generalized method of averaging, because the error in so doing is assumed small. It will be shown numerically in the next section that this approach yields a tractable solution which is a good first-order approximation for a varied class of satellite orbits.

The theoretical framework for the (MAV) solution is given in [4], and in this section the procedure will be outlined with some technical details noted. The actual solution produced in the appendix of [4] is given for just the $l = 2$ harmonic in (1.2) while the solution in the appendix of this report corrects some minor algebraic errors and also includes the $l = 3$ harmonic. This has proved necessary for high-altitude satellites such as those in synchronous or half-synchronous orbits. The (MAV) solution involves an enormous amount of algebra, especially in averaging over the mean anomaly of the Moon. This is a mathematical disadvantage, since there is no way to rely on the work which produced the $l = 2$ solution for the $l = 3$ solution as in the (SEM) case. Because of the bulky amount of algebra involved in the (MAV) solution, computer algebra is very helpful in performing the integration since it can produce fast, error-free computations with FORTRAN code also available. The solution presented in the appendix was derived by MIT's MACSYMA symbolic manipulation program and a FORTRAN implementation on magnetic tape of either the (SEM) solution or the (MAV) solution is available from the author upon request.

If $\dot{\mathbf{z}}$ represents a vector containing the six Lagrange differential equations describing the motion of the vector of satellite elements $\mathbf{z}$ under the disturbing force of the Moon (1.2), the (MAV) solution is started with an average of $\dot{\mathbf{z}}$ over the mean anomaly of the satellite

$$\langle \dot{\mathbf{z}} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \dot{\mathbf{z}} \mathrm{dM}. \tag{3.1}$$

In the case at hand, it is convenient to use the eccentric anomaly E to compute the indefinite integral

$$I_1 = \int \dot{\mathbf{z}}(E)(1 - e\cos E)\mathrm{dE}$$

so that the short period corrections appear as a by-product

$$n\delta\mathbf{z}_s = I_1 - E\langle \dot{\mathbf{z}} \rangle,$$

where n denotes the mean motion of the satellite. Since this is an initial value problem, we can remove the constant part of this quantity. A second averaging is now performed on $\langle \dot{\mathbf{z}} \rangle$ with respect to the mean anomaly of the Moon

$$\langle\langle \dot{\mathbf{z}} \rangle\rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle \dot{\mathbf{z}} \rangle \mathrm{dM_k}.$$

and this is calculated by converting $\langle\dot{\mathbf{z}}\rangle$ in terms of the true anomaly of the Moon and computing the indefinite integral

$$I_2 = \int \langle\dot{\mathbf{z}}\rangle(f_k)\frac{(1-e_k^2)^{3/2}}{(1+e_k\cos f_k)^2}\mathrm{df_k}.$$

The long period corrections are readily available as

$$n_k\delta\mathbf{z}_l = I_2 - f_k\langle\langle\dot{\mathbf{z}}\rangle\rangle,$$

where n_k denotes the mean motion of the Moon. Thus to update given mean elements $\mathbf{z}_0$ (at a given epoch t_0) to a desired time t, one propagates the mean elements

$$\mathbf{z}_1 = \mathbf{z}_0 + \langle\langle\dot{\mathbf{z}}\rangle\rangle(t - t_0),$$

adds the long period corrections

$$\mathbf{z}_2 = \mathbf{z}_1 + \delta\mathbf{z}_l(\mathbf{z}_1),$$

and then adds the short period corrections

$$\mathbf{z}(t) = \mathbf{z}_2 + \delta\mathbf{z}_s(\mathbf{z}_2).$$

Since the equation for the mean anomaly rate $\dot{M}$ has the mean motion n included and $\langle\dot{a}\rangle = 0$, it is necessary in a first-order development to include

$$-\frac{3}{2}\frac{n\delta a_s}{a}$$

in the equation of $\dot{M}$ before any averaging of this equation is performed.

Some technical details involved in the averaging mentioned above will now be presented. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ denote a fixed Earth centered inertial coordinate system referred to the equator (with $\mathbf{i}$ lying in the ecliptic plane pointing to a vernal equinox) and let $\mathbf{z}_0$ be a vector of the six initial Kepler elements referred to this system. Now, let $\mathbf{P}$ denote the unit vector pointing to perigee of the elliptical satellite orbit and let $\mathbf{Q} = \frac{\partial \mathbf{P}}{\partial \omega}$ denote the unit vector orthogonal to $\mathbf{P}$ in the satellite plane. To complete a right-hand system, let $\mathbf{u}_N$ denote the unit normal to the satellite plane, $\mathbf{u}_N \equiv \mathbf{P}\times\mathbf{Q}$. It is well known that

$$\begin{aligned}\mathbf{P} \;=\; & (\cos\Omega\cos\omega - \cos I\sin\Omega\sin\omega)\mathbf{i} + \\ & (\sin\Omega\cos\omega + \cos I\cos\Omega\sin\omega)\mathbf{j} + \\ & (\sin I\sin\omega)\mathbf{k} \end{aligned} \tag{3.2}$$

and

$$\mathbf{u}_N = (\sin\Omega\sin I)\mathbf{i} - (\cos\Omega\sin I)\mathbf{j} + (\cos I)\mathbf{k}. \tag{3.3}$$

The unit vector $\mathbf{1}_r$ pointing to the satellite at any time can be written

$$\mathbf{1}_r = \cos(f)\mathbf{P} + \sin(f)\mathbf{Q},$$

and if $\mathbf{1}_k$ denotes a unit vector pointing to the Moon, for the first averaging it suffices to use the projections of $\mathbf{1}_k$ onto $\mathbf{P}, \mathbf{Q}$, and $\mathbf{u}_N$. These projections will be denoted $S1$, $S2$, and $S3$, respectively, so that we can express

$$\cos\psi_k = \mathbf{1}_r \cdot \mathbf{1}_k = \cos(f)S1 + \sin(f)S2.$$

The projection $S3 = \mathbf{u}_N \cdot \mathbf{1}_k$ appears in the Lagrange equations for the rates of the argument of perigee $\dot{\omega}$, right ascending node $\dot{\Omega}$, and the inclination $\dot{I}$ since $\frac{\partial \mathbf{P}}{\partial I} = \mathbf{u}_N \sin\omega$ and $\cos I \frac{\partial \mathbf{Q}}{\partial \omega} - \frac{\partial \mathbf{Q}}{\partial \Omega} = -\sin I \frac{\partial \mathbf{P}}{\partial I}$.

For the second averaging over the mean anomaly of the Moon, some geometrical transformations are needed. From Kepler elements for the Moon's orbit, we can compute $\mathbf{u}_k$, the unit normal of the Moon's orbital plane, and $\mathbf{P}_k$, the vector pointing to the Moon's perigee, as in (3.3) and (3.2), respectively. If the elements for the Moon's orbit are referred to the ecliptic, then an additional transformation must be made to place these vectors into the inertial reference frame $(\mathbf{i}, \mathbf{j}, \mathbf{k})$. This is merely a plane rotation about the ecliptic inclination ι; for if $\mathbf{v} = c_1\mathbf{i}_\iota + c_2\mathbf{j}_\iota + c_3\mathbf{k}_\iota$ is any vector referred to an inertial system about the ecliptic plane (with $\mathbf{i}_\iota$ coinciding with $\mathbf{i}$), then we can write $\mathbf{v} = c_1\mathbf{i} + (c_2\cos\iota - c_3\sin\iota)\mathbf{j} + (c_2\sin\iota + c_3\cos\iota)\mathbf{k}$ in the inertial reference frame about the equator. As in the case of the satellite orbit, if we define $\mathbf{Q}_k \equiv \mathbf{u}_k \times \mathbf{P}_k$, then we can express

$$\mathbf{1}_k = \cos(f_k)\mathbf{P}_k + \sin(f_k)\mathbf{Q}_k.$$

It is now convenient to write the inclination of the Moon with respect to the satellite plane

$$\cos jk \equiv \mathbf{u}_N \cdot \mathbf{u}_k,$$

and compute the unit vector pointing in the direction of the nodal line which intersects the plane of the satellite orbit and the plane of the lunar orbit,

$$\mathbf{u}_L \equiv \frac{\mathbf{u}_k \times \mathbf{u}_N}{\sin jk}.$$

To complete a right-hand system for this representation of the satellite plane, we define

$$\mathbf{u}_{NL} \equiv \mathbf{u}_N \times \mathbf{u}_L$$

so that

$$\mathbf{P} = \cos(apk)\mathbf{u}_L + \sin(apk)\mathbf{u}_{NL}$$

and

$$\mathbf{Q} = -\sin(apk)\mathbf{u}_L + \cos(apk)\mathbf{u}_{NL},$$

where apk is the argument of perigee of the satellite orbit with respect to the Moon's orbit. To complete a right-hand system for this representation of the lunar orbit, we define

$$\mathbf{u}_{kL} \equiv \mathbf{u}_k \times \mathbf{u}_L$$

so that

$$\mathbf{P}_k = -\cos(wk)\mathbf{u}_L - \sin(wk)\mathbf{u}_{kL}$$

and

$$\mathbf{Q}_k = \sin(wk)\mathbf{u}_L - \cos(wk)\mathbf{u}_{kL},$$

where wk is the argument of perigee of the lunar orbit measured with respect to the satellite orbit. Note that the nodal vector $\mathbf{u}_L$ points to the descending node of this system for the Moon but points to the ascending node of the system for the satellite.

It is not difficult to use this information to represent $S1$, $S2$, and $S3$ in their final form so that an averaging over the mean anomaly of the Moon can be performed:

$$\begin{aligned}
S1 &= -\cos(apk)\cos(wk + f_k) - \sin(apk)\cos(jk)\sin(wk + f_k) \\
S2 &= \sin(apk)\cos(wk + f_k) - \cos(apk)\cos(jk)\sin(wk + f_k) \\
S3 &= \sin(jk)\sin(wk + f_k).
\end{aligned}$$

Of course, all of the equations carry over verbatim for use with the solar disturbing function with appropriate consideration for the values pertaining to the Sun's force in place of those for the Moon's force.

4. NUMERICAL COMPARISONS

The (SEM) and (MAV) solutions have been coded in FORTRAN for a Harris 800 series computer with double precision (48-bit) arithmetic. These solutions will be compared numerically in two settings. One is a comparison of each analytical solution against a straight numerical integration of the Lagrange equations of motion, and the second is a test to see how well each solution performs in an analytical orbit determination routine which fits (in the weighted least squares sense) observed data to a model of propagation. The observed data in this case are a set of topocentric sites pointing to an object consisting of azimuth, elevation, range, and range rate. The data were simulated by a special perturbations precision numerical integration routine which integrates Newton's law of gravitation directly from (1.1) (and therefore requires no truncation of a Legendre expansion). Simulated data were chosen over real data for the following reasons: simulated data were essentially free of noise, other forces can be turned on or off as desired so that interactions can be investigated, and all residuals are directly related to the analytical model instead of possibly belonging to some unknown effect in the data. An additional advantage to using simulated data were that various ephemeris models for the Moon can be placed into the integration such as the simple mean Kepler theory (which is assumed by the analytical theories), a more accurate and complex lunar theory such as the Hill-Brown model [12], or interpolation from ephemeris tapes which have been provided by the U.S. Naval Observatory. In any case, all that is needed in a numerical integration of the equations of motion is a unit vector pointing to the Moon together with its range at any desired time.

All of the tests below involve a generic geosynchronous object with parameters: $a = 42,164.1$ km, $e = .001$, $I = 1.5\,\text{deg}$, $\omega = 145\,\text{deg}$, $\Omega = 166\,\text{deg}$, and $M = 25.413\,\text{deg}$ at the epoch day 280.28046 of 1986. Considering the present-day satellite population, the effect of the Moon's gravitational force is most significant at synchronous altitudes, with the obvious exception of super-synchronous objects with high eccentricity such as the Soviet Prognoz series. Such objects reach over half of the distance to the Moon at apogee, and to use this class of satellites in numerical experiments with the analytical theories in question would be pointless because a truncation of the force model to just the second and third harmonics is no longer justified. On the other hand, a geosynchronous object is ideal for numerical comparisons of this nature because the perturbations are significant, the eccentricity is small (so that a truncation of the Hansen coefficients in the (SEM) solution can be chosen at $|\, j \,| \leq 3$), the third harmonic plays a noticeable role in achieving a desired accuracy, and the class of synchronous objects is one of the largest classes of active satellites most affected by the Moon's gravitational force.

First, the (SEM) and (MAV) solutions were compared to a fifth- and sixth-order variable step size Runge-Kutta numerical integration. Recall that both of the analytical solutions assume that the ephemeris for the Moon is described by a Kepler ellipse with constant rates of change for the angular quantities. In the numerical integration, it is possible to insert any ephemeris model for the Moon. We will use two different ephemeris models for comparison in this experiment. The Kepler ephemeris which is used for the analytical solutions will be referred to as the "simple model," and a more complicated and accurate model known as the Hill-Brown lunar theory [12] will be referred

to as the "Hill-Brown model." The purpose in this is that it allows one to see the effect in pure propagation of simplifying the ephemeris model so that integration is possible. In each of the plots which follow, the effect upon the semimajor axis and mean longitude ($\delta\lambda = \delta\omega + \delta M + \delta\Omega$) is displayed.

Figure 4-1 displays the total perturbation in a and λ from the Moon's direct gravitational force. Both the second and third harmonics were used in the force model and the simple lunar ephemeris model was used in the numerical integration. Notice that the amplitude in the change to the semimajor axis reaches 2 km and that the perturbation of the mean longitude has a strong secular rate. It was observed (but not plotted here) that the amplitude of the change in the semimajor axis from the second harmonic in the force model was approximately 2 km, the amplitude from the third harmonic in the force model was approximately 300 m, and the amplitude from the fourth harmonic in the force model was approximately 20 m. The comparison of this graph to the (SEM) solution is shown in Figure 4-2, and the comparison to the (MAV) solution is shown in Figure 4-3. It appears that under these special circumstances, the (SEM) solution exhibits smaller error residuals by an order of magnitude. The errors in the (SEM) solution behave as one would expect from a first-order theory (about 10^{-3} to 10^{-4} of the perturbation), but the (MAV) solution has larger errors in the short period corrections. These differences reflect upon the approximation to the true solution for each theory, however, they become less pronounced when the Hill-Brown ephemeris is used in the numerical integration. The error in the (SEM) solution for this situation is displayed in Figure 4-4, where one can now see the amplitude of the error in the semimajor axis reaching 100 m.

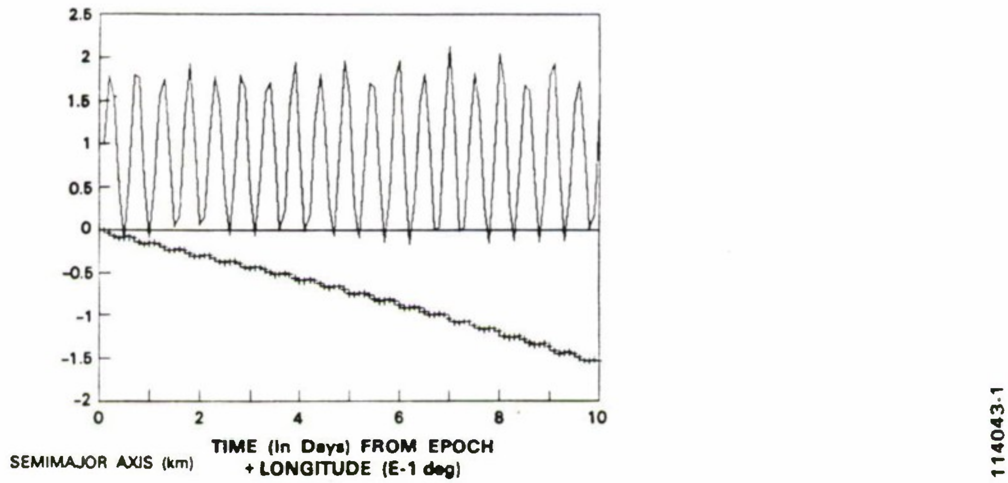
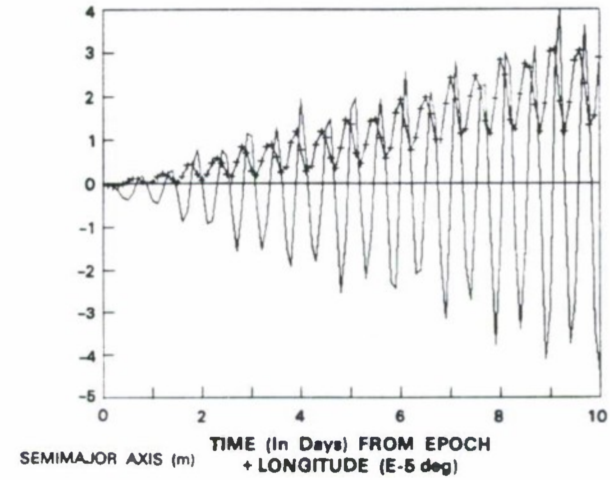


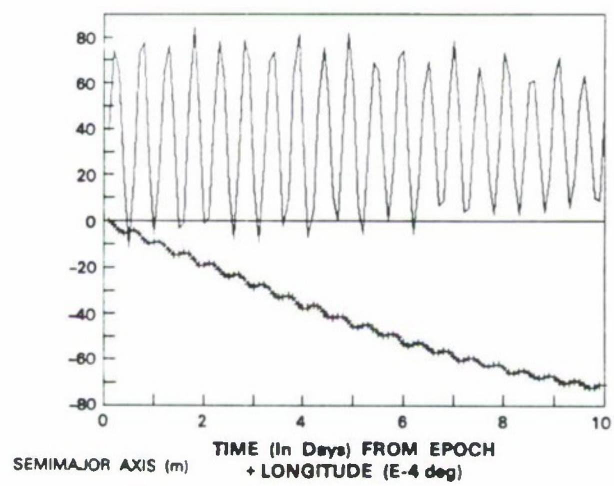
Figure 4-1. The lunar perturbation of the semimajor axis and mean longitude.

The length of propagation is another parameter which may tell more about the effective accuracy of the two solutions. In Figures 4-5 and 4-6, the error in the propagation of the semimajor



114043-2

Figure 4-2. The (SEM) error in the semimajor axis and the mean longitude with the simple lunar ephemeris model used in the numerical reference.



114043-3

Figure 4-3. The (MAV) error in the semimajor axis and the mean longitude with the simple lunar ephemeris model used in the numerical reference.

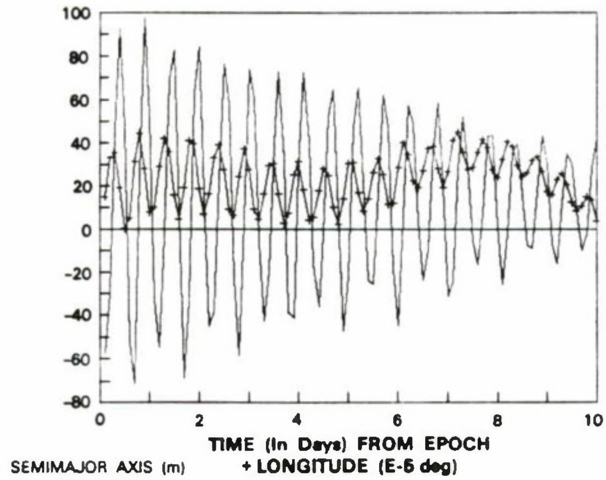


Figure 4-4. The (SEM) error in the semimajor axis and the mean longitude with the Hill-Brown lunar ephemeris model used in the numerical reference.

axis and mean longitude is extended to 100 days from the epoch for the (SEM) solution and the (MAV) solution, respectively. This is for comparison with the plots in Figures 4-2 and 4-3, and so the simple lunar ephemeris model was used in the numerical integration. Incidentally, the integration constants, which are different for both theories, contribute to the error growth and apparent biases displayed in these plots. This is of no great concern since when each theory is fit to observed data, these constants are automatically fit in the process. These plots show that even over 100 revolutions of the satellite orbit, both theories are able to predict the lunar perturbations to within reasonable tolerances.

In many cases, an analytical theory is chosen so that speed in computation can be achieved for routine calculations. Often, near real time experiments are desired, and precision numerical theories are not able to meet this requirement. Thus, it is necessary to consider computation speed as an important parameter in order to determine the usefulness of an analytical theory. Table 4-1 summarizes the time (in milliseconds) required for one propagation from each solution on the Harris H-800 computer, where the force model included just the second harmonic, or the second and the third harmonics. It is seen that the (MAV) solution is faster by an order of magnitude. This is not surprising since a rough count of the number of terms in (2.7) indicates that there are $3 \times 3 \times 3 \times 3 \times 7 \times 7 = 3,969$ periodic terms in the expanded force model for just the $l = 2$ Legendre polynomial, and for the $l = 2$ and $l = 3$ force model, there are $3,969 + 4 \times 4 \times 4 \times 4 \times 7 \times 7 = 16,513$ periodic terms. Some of these terms can be neglected in a "smart" implementation of the theory because they contribute little to the overall expansion, and clever bookkeeping is required to optimize the running time. However, it is clear that this solution contains a lot of terms and requires

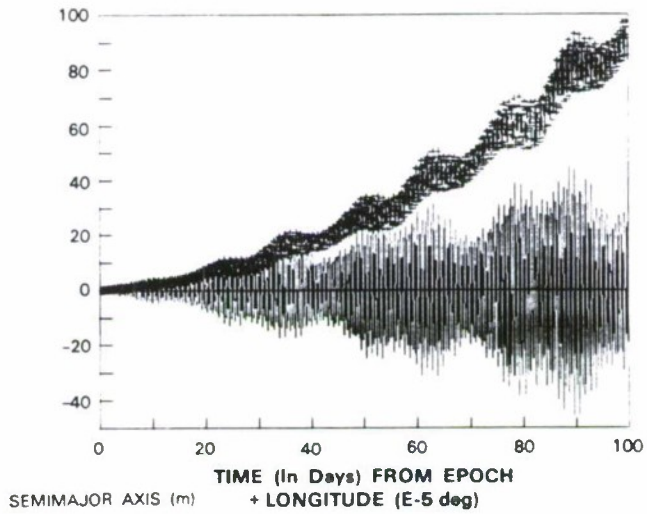


Figure 4-5. *The (SEM) error in the semimajor axis and the mean longitude with the simple lunar ephemeris model used in the numerical reference.*

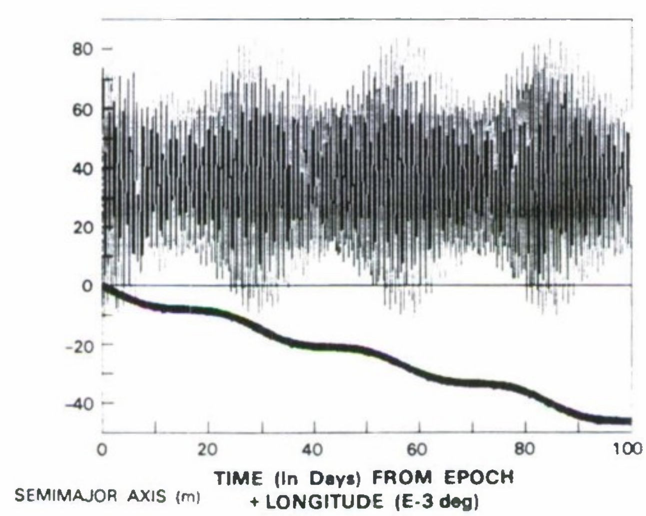


Figure 4-6. *The (MAV) error in the semimajor axis and the mean longitude with the simple lunar ephemeris model used in the numerical reference.*

the evaluation of hundreds (and possibly thousands) of sines and cosines. While the (MAV) solution can also have thousands of terms, there are only a handful of different trigonometric functions to evaluate, and this causes the overall running time for this solution to be much shorter than for the (SEM) solution (since multiplication and addition take less time to compute than the evaluation of transcendental functions).

TABLE 4-1.

Computation Time Required for One Propagation

Force Model	(SEM) solution	(MAV) solution
$l = 2$ harmonic only	85 ms	11 ms
$l = 2$ and $l = 3$ harmonics	256 ms	27 ms

The final numerical experiment involves fitting mean elements to numerically simulated data using the (SEM) solution or the (MAV) solution as a propagator. Ten days of observed data were generated using precision numerical integration of the special perturbations from the lunar force model. The data were propagated directly from (1.1), and therefore, no expansion (or truncation) of Legendre polynomials was employed. Two sets of data consisting of azimuth, elevation, range. and range rate from a selected site location were generated, with the first using the simple ephemeris model for the Moon and the second using the more realistic Hill-Brown ephemeris for the Moon. The converged residuals are presented in Table 4-2. It is interesting to note that, in this experiment. there is very little difference in accuracy between the two analytical theories. It is also clear that a mean or osculating ephemeris for the Moon is not a disturbing factor in this experiment (as is expected since this is an iterative process). The third row in Table 4-2 is presented only to show the large residuals in range which result when the third harmonic is omitted from the force model for this particular object. The accuracy in range is degraded by over 300 m.

In spite of the results from the direct propagation experiment, the fitting experiment and the time computation experiment demonstrate that the (MAV) solution is more practical and just as accurate for most purposes. The assumptions provide sufficient accuracy to within 70 to 100 m in range and 2 mdeg in azimuth and elevation, and it is doubtful that these levels can be reduced by any addition to the theory.

TABLE 4-2.

Converged rms Residuals as Compared Against Ephemeris Model in the Reference and Force Model in the Solution

(MAV)/(SEM)	Az. (mdeg)	El. (mdeg)	r (m)	$dotr$ (cm/s)
Simple ephemeris model $l = 2, 3$ harmonics	1 / 1	1 / 1	70 / 80	1 / 1
Hill-Brown model $l = 2, 3$ harmonics	1 / 1	1 / 1	60 /100	1 / 1
Hill-Brown model $l = 2$ harmonic	2 / 2	1 / 1	420 / 420	3 / 3

5. SUMMARY

Two analytical solutions to the problem of describing the direct lunar gravitational effect upon artificial satellite orbits have been presented and compared from both a theoretical and numerical viewpoint. The (SEM) solution uses traditional expansions about the inclination and eccentricity of both the satellite and the Moon so that a simultaneous integration of both motions is afforded. while the (MAV) solution employs the method of averaging on the unexpanded disturbing function (1.2) so that the motions of the satellite and the Moon can be integrated separately. The (SEM) solution is complete for all Legendre harmonics in the disturbing function (1.2), whereas the (MAV) solution is complicated to express for more than the second and third harmonics. The following conclusions can be drawn.

- The (SEM) solution is more pleasing mathematically because it provides a compact expression for the complete problem, while the (MAV) solution presents a bulky package of equations for one harmonic at a time.

- The simultaneous integration exhibited by the (SEM) solution achieves about an order of magnitude (or more) in accuracy over the (MAV) solution when comparing the perturbation of the Kepler elements of the satellite against a numerical integration of the Lagrange equations of motion with an ephemeris for the Moon approximated by mean Kepler motion. Under these conditions, a ten-day integration of a geosynchronous object exhibits an error of about 3 m and 70 m for the semimajor axis from the (SEM) solution and the (MAV) solution. respectively. The error in mean longitude is approximately 10^{-5} deg for the (SEM) solution and 10^{-4} deg for the (MAV) solution. The total perturbation in this experiment reached 2 km for the semimajor axis and 10^{-1} deg for the mean longitude.

- The (SEM) solution can be more than an order of magnitude slower in computation speed for one propagation than the (MAV) solution, and if near real time execution is necessary, then the (SEM) solution is in serious jeopardy of being entirely impractical for this purpose. Moreover, the execution time required for the (SEM) solution increases significantly for objects with high eccentricity, while the (MAV) solution shows no increase for such objects.

- Both the (SEM) and (MAV) solutions perform the same when asked to fit simulated data for a geosynchronous object over a ten-day period. Converged residuals were on the order of a millidegree in azimuth and elevation, 70 m in range, and 10 mm/s in range rate.

- The ephemeris model has little effect on the converged residuals for both models, and this suggests that, for this application, the assumption of mean Kepler motion for the Moon in the analytical theories is good for a first-order theory.

- Both the second and third harmonics in the expansion of the disturbing function (1.2) are necessary for high-altitude orbits in order to obtain an accuracy of less than 100 m in range.

APPENDIX A

The method of averaging solution for the $l = 2$ and $l = 3$ harmonics in the disturbing function is presented below. The solution was produced and checked by MACSYMA, a computer algebra manipulation program developed at MIT. The solution should be, for the most part, free from errors (although the process of transferring the solution to paper can involve human error). The equations which are reproduced below have been coded in FORTRAN (also by MACSYMA) and numerical tests indicate that the the equations are correct. The solution is written in terms of quantities already defined except for combinations of magnitude which will be defined below.

The first averaging is over the mean anomaly of the satellite,and for the $l = 2$ harmonic we define

$$za \equiv n\left(\frac{Gm_k}{\mu}\right)\left(\frac{a}{r_k}\right)^3,$$

where $\frac{Gm_k}{\mu}$ is the ratio of the mass of the Moon to the mass of the Earth. For the $l = 3$ harmonic, we define

$$za2 \equiv za\left(\frac{a}{r_k}\right).$$

The result of averaging the $l = 2$ harmonic over the mean anomaly of the satellite yields

$$\langle \dot{a} \rangle_2 = 0$$

$$\langle \dot{I} \rangle_2 =$$

```
          2                            2
3 s3 (e  s2 sin(w) - s2 sin(w) + 4 e  s1 cos(w) + s1 cos(w)) za
---------------------------------------------------------------
                                     2 1/2
                           2 (1 - e )
```

$$\langle\dot{\Omega}\rangle_2 =$$

$$\frac{3\, s3\, (4 e^2\, s1 \sin(w) + s1 \sin(w) - e^2\, s2 \cos(w) + s2 \cos(w))\, za}{2\, (1 - e^2)^{1/2} \sin(i)}$$

$$\langle\dot{\epsilon}\rangle_2 =$$

$$- \frac{15\, e\, (1 - e^2)^{1/2}\, s1\, s2\, za}{2}$$

$$\langle\dot{\omega}\rangle_2 = -\cos I \langle\dot{\Omega}\rangle_2 +$$

$$- \frac{3\, (1 - e^2)^{1/2}\, (s2^2 - 4\, s1^2 + 1)\, za}{2}$$

$$\langle\dot{M} - n_0\rangle_2 = -\sqrt{1 - \epsilon^2}\left[\langle\dot{\omega}\rangle_2 + \cos I \langle\dot{\Omega}\rangle_2\right] +$$

$$\frac{(6\, e^2\, s2^2 - 6\, s2^2 - 33\, e^2\, s1^2 - 6\, s1^2 + 9\, e^2 + 4)\, za}{2}$$

with short period corrections

$$n\delta a_2 =$$

$$\begin{aligned}
&- a\, (6\, e^2\, s1\, s2 \sin(2\, E) - 6\, s1\, s2 \sin(2\, E) \\
&- 3\, e^2\, (1 - e^2)^{1/2}\, s2^2 \cos(2\, E) + 3\, (1 - e^2)^{1/2}\, s2^2 \cos(2\, E) \\
&- 3\, (1 - e^2)^{1/2}\, s1^2 \cos(2\, E) + e^2\, (1 - e^2)^{1/2} \cos(2\, E) - 12\, e^3\, s1\, s2 \sin(E) \\
&+ 12\, e\, s1\, s2 \sin(E) + 12\, e\, (1 - e^2)^{1/2}\, s1^2 \cos(E) - 4\, e\, (1 - e^2)^{1/2} \cos(E))\, za \\
&/(2\, (1 - e^2)^{1/2})
\end{aligned}$$

$n\delta I_2 =$

$$
\begin{aligned}
& - s3 \, (e^3 (1 - e^2)^{1/2} \, s2 \sin(3 E) \sin(w) \\
& - e (1 - e^2)^{1/2} \, s2 \sin(3 E) \sin(w) + e^3 \, s1 \cos(3 E) \sin(w) \\
& - e \, s1 \cos(3 E) \sin(w) - 3 e^2 (1 - e^2)^{1/2} \, s2 \sin(2 E) \sin(w) \\
& + 3 (1 - e^2)^{1/2} \, s2 \sin(2 E) \sin(w) - 3 e^4 \, s1 \cos(2 E) \sin(w) \\
& + 3 \, s1 \cos(2 E) \sin(w) - 3 e^3 (1 - e^2)^{1/2} \, s2 \sin(E) \sin(w) \\
& + 3 e (1 - e^2)^{1/2} \, s2 \sin(E) \sin(w) + 15 e^3 \, s1 \cos(E) \sin(w) \\
& - 15 e \, s1 \cos(E) \sin(w) - e (1 - e^2)^{1/2} \, s1 \sin(3 E) \cos(w) \\
& - e^3 \, s2 \cos(3 E) \cos(w) + e \, s2 \cos(3 E) \cos(w) \\
& + 6 e^2 (1 - e^2)^{1/2} \, s1 \sin(2 E) \cos(w) + 3 (1 - e^2)^{1/2} \, s1 \sin(2 E) \cos(w) \\
& + 3 e^4 \, s2 \cos(2 E) \cos(w) - 3 \, s2 \cos(2 E) \cos(w) \\
& - 12 e^3 (1 - e^2)^{1/2} \, s1 \sin(E) \cos(w) - 33 e (1 - e^2)^{1/2} \, s1 \sin(E) \cos(w) \\
& - 15 e^3 \, s2 \cos(E) \cos(w) + 15 e \, s2 \cos(E) \cos(w)) \, za/(4 (e - 1) (e + 1))
\end{aligned}
$$

$n\delta \Omega_2 =$

$$
\begin{aligned}
& s3 \, (e (1 - e^2)^{1/2} \, s1 \sin(3 E) \sin(w) + e^3 \, s2 \cos(3 E) \sin(w) \\
& - e \, s2 \cos(3 E) \sin(w) - 6 e^2 (1 - e^2)^{1/2} \, s1 \sin(2 E) \sin(w) \\
& - 3 (1 - e^2)^{1/2} \, s1 \sin(2 E) \sin(w) - 3 e^4 \, s2 \cos(2 E) \sin(w) \\
& + 3 \, s2 \cos(2 E) \sin(w) + 12 e^3 (1 - e^2)^{1/2} \, s1 \sin(E) \sin(w) \\
& + 33 e (1 - e^2)^{1/2} \, s1 \sin(E) \sin(w) + 15 e^3 \, s2 \cos(E) \sin(w) \\
& - 15 e \, s2 \cos(E) \sin(w) + e^3 (1 - e^2)^{1/2} \, s2 \sin(3 E) \cos(w) \\
& - e (1 - e^2)^{1/2} \, s2 \sin(3 E) \cos(w) + e^3 \, s1 \cos(3 E) \cos(w) \\
& - e \, s1 \cos(3 E) \cos(w) - 3 e^2 (1 - e^2)^{1/2} \, s2 \sin(2 E) \cos(w) \\
& + 3 (1 - e^2)^{1/2} \, s2 \sin(2 E) \cos(w) - 3 e^4 \, s1 \cos(2 E) \cos(w) \\
& + 3 \, s1 \cos(2 E) \cos(w) - 3 e^3 (1 - e^2)^{1/2} \, s2 \sin(E) \cos(w) \\
& + 3 e (1 - e^2)^{1/2} \, s2 \sin(E) \cos(w) + 15 e^3 \, s1 \cos(E) \cos(w) \\
& - 15 e \, s1 \cos(E) \cos(w)) \, za/(4 (e - 1) (e + 1) \sin(i))
\end{aligned}
$$

$$n\delta e_2 = \frac{1-e^2}{2ae} n\delta a_2 +$$

```
             3         2 1/2                              2 1/2
     - (e   (1 - e )     s1 s2 sin(3 E) - 2 e (1 - e )     s1 s2 sin(3 E)
    3   2                   2               3   2                 2
 - e  s2  cos(3 E) + e s2  cos(3 E) + e  s1  cos(3 E) - e s1  cos(3 E)
     2       2 1/2                           2 1/2
 + 3 e  (1 - e )    s1 s2 sin(2 E) + 6 (1 - e )    s1 s2 sin(2 E)
      4   2                2                4   2                 2
 + 3 e  s2  cos(2 E) - 3 s2  cos(2 E) - 3 e  s1  cos(2 E) + 3 s1  cos(2 E)
       3       2 1/2                             2 1/2
 - 15 e  (1 - e )    s1 s2 sin(E) - 30 e (1 - e )    s1 s2 sin(E)
       3   2                   2            3   2                 2
 - 15 e  s2  cos(E) + 15 e s2  cos(E) + 15 e  s1  cos(E) - 15 e s1  cos(E)) za

/(4 e)
```

$$n\delta\omega_2 = -\cos(I) n\delta\Omega_2 +$$

```
                  2 1/2   2                          2 1/2   2
     - ((1 - e )     s2  sin(3 E) - (1 - e )     s1  sin(3 E)
    2                                                2 1/2   2
 - e  s1 s2 cos(3 E) + 2 s1 s2 cos(3 E) - 3 e (1 - e )     s2  sin(2 E)
            2 1/2                3
 + e (1 - e )     sin(2 E) - 3 e  s1 s2 cos(2 E) - 3 e s1 s2 cos(2 E)
           2 1/2   2             2        2 1/2   2
 - 3 (1 - e )     s2  sin(E) + 12 e  (1 - e )     s1  sin(E)
            2 1/2   2            2        2 1/2                  2 1/2
 + 15 (1 - e )     s1  sin(E) - 4 e  (1 - e )     sin(E) - 4 (1 - e )     sin(E)
      2
 + 33 e  s1 s2 cos(E) - 18 s1 s2 cos(E)) za/(4 e)
```

$$n\delta M_2 = -\sqrt{1-e^2}\left[n\delta\omega_2 + \cos(I) n\delta\Omega_2\right] +$$

```
             3        2 1/2   2                              2 1/2   2
     (21 e   (1 - e )     s2  sin(3 E) - 21 e (1 - e )     s2  sin(3 E)
              2 1/2   2                 3        2 1/2
 + 21 e (1 - e )     s1  sin(3 E) - 7 e  (1 - e )     sin(3 E)
       3                                         2       2 1/2   2
 + 42 e  s1 s2 cos(3 E) - 42 e s1 s2 cos(3 E) - 63 e  (1 - e )     s2  sin(2 E)
            2 1/2   2                 2       2 1/2   2
 + 63 (1 - e )     s2  sin(2 E) - 126 e  (1 - e )     s1  sin(2 E)
            2 1/2   2                2        2 1/2
 - 63 (1 - e )     s1  sin(2 E) + 63 e  (1 - e )     sin(2 E)
       4                     2
 - 72 e  s1 s2 cos(2 E) - 54 e  s1 s2 cos(2 E) + 126 s1 s2 cos(2 E)
      3       2 1/2   2                      2 1/2   2
 - 9 e  (1 - e )     s2  sin(E) + 9 e (1 - e )     s2  sin(E)
        3       2 1/2   2                        2 1/2   2
 + 144 e  (1 - e )     s1  sin(E) + 639 e (1 - e )     s1  sin(E)
       3       2 1/2                         2 1/2                  4           2
 - 45 e  (1 - e )     sin(E) - 216 e (1 - e )     sin(E) - 108 e  s1 s2 cos (E)
```

```
              2             2            3
+ 108 e  s1 s2 cos (E) + 630 e  s1 s2 cos(E) - 630 e s1 s2 cos(E)
      4            2                   2 1/2
+ 54 e  s1 s2 - 54 e  s1 s2) za/(24 (1 - e )    )
```

The result of averaging the $l = 3$ harmonic over the mean anomaly of the satellite yields

$$\langle \dot{a} \rangle_3 = 0$$

$$\langle \dot{I} \rangle_3 =$$

```
                         2                                    2   2
      - 15 e s3 (10 e  s1 s2 sin(w) - 10 s1 s2 sin(w) - 5 e  s2  cos(w)
        2                2   2                2               2
+ 5 s2  cos(w) + 20 e  s1  cos(w) + 15 s1  cos(w) - 3 e  cos(w) - 4 cos(w))
                  2 1/2
za2/(16 (1 - e )    )
```

$$\langle \dot{\Omega} \rangle_3 =$$

```
                     2   2                 2                  2   2
      15 e s3 (5 e  s2  sin(w) - 5 s2  sin(w) - 20 e  s1  sin(w)
        2                2                           2
- 15 s1  sin(w) + 3 e  sin(w) + 4 sin(w) + 10 e  s1 s2 cos(w)
                                         2 1/2
- 10 s1 s2 cos(w)) za2/(16 (1 - e )    sin(i))
```

$$\langle \dot{e} \rangle_3 =$$

```
                 2 1/2          2   2         2          2   2         2        2
       15 (1 - e )     s2 (5 e  s2   - 5 s2  - 30 e  s1   - 5 s1  + 3 e  + 4) za2
     - -------------------------------------------------------------------------
                                              16
```

$$\langle \dot{\omega} \rangle_3 = -\cos I \langle \dot{\Omega} \rangle_3 +$$

```
                2 1/2          2   2          2          2   2         2        2
      15 (1 - e )     s1 (15 e  s2   - 5 s2  - 20 e  s1   - 5 s1  + 9 e  + 4) za2
      -------------------------------------------------------------------------
                                             16 e
```

$$\langle \dot M - n_0 \rangle_3 = -\sqrt{1-e^2}\left[\langle\dot\omega\rangle_3 + \cos I \langle\dot\Omega\rangle_3\right] +$$

$$- \frac{3\, e\, s1\, (165\, e^2\, s2^2 - 165\, s2^2 - 260\, e^2\, s1^2 - 165\, s1^2 + 123\, e^2 + 132)\, za2}{16}$$

with short period corrections

$$n\delta a_3 =$$

$$\begin{aligned}
&- a\, (5\, e^4\, s2^3\, \sin(3\,E) - 10\, e^2\, s2^3\, \sin(3\,E) + 5\, s2^3\, \sin(3\,E) \\
&+ 15\, e^2\, s1^2\, s2\, \sin(3\,E) - 15\, s1^2\, s2\, \sin(3\,E) - 3\, e^4\, s2\, \sin(3\,E) \\
&+ 3\, e^2\, s2\, \sin(3\,E) - 15\, e^2\, (1 - e^2)^{1/2}\, s1\, s2^2\, \cos(3\,E) \\
&+ 15\, (1 - e^2)^{1/2}\, s1\, s2^2\, \cos(3\,E) - 5\, (1 - e^2)^{1/2}\, s1^3\, \cos(3\,E) \\
&+ 3\, e^2\, (1 - e^2)^{1/2}\, s1\, \cos(3\,E) - 60\, e^3\, s1^2\, s2\, \sin(2\,E) + 60\, e\, s1^2\, s2\, \sin(2\,E) \\
&+ 12\, e^3\, s2\, \sin(2\,E) - 12\, e\, s2\, \sin(2\,E) + 30\, e^3\, (1 - e^2)^{1/2}\, s1\, s2^2\, \cos(2\,E) \\
&- 30\, e\, (1 - e^2)^{1/2}\, s1\, s2^2\, \cos(2\,E) + 30\, e\, (1 - e^2)^{1/2}\, s1^3\, \cos(2\,E) \\
&- 6\, e^3\, (1 - e^2)^{1/2}\, s1\, \cos(2\,E) - 12\, e\, (1 - e^2)^{1/2}\, s1\, \cos(2\,E) \\
&- 15\, e^4\, s2^3\, \sin(E) + 30\, e^2\, s2^3\, \sin(E) - 15\, s2^3\, \sin(E) + 60\, e^4\, s1^2\, s2\, \sin(E) \\
&- 45\, e^2\, s1^2\, s2\, \sin(E) - 15\, s1^2\, s2\, \sin(E) - 3\, e^4\, s2\, \sin(E) - 9\, e^2\, s2\, \sin(E) \\
&+ 12\, s2\, \sin(E) + 15\, e^2\, (1 - e^2)^{1/2}\, s1\, s2^2\, \cos(E) \\
&- 15\, (1 - e^2)^{1/2}\, s1\, s2^2\, \cos(E) - 60\, e^2\, (1 - e^2)^{1/2}\, s1^3\, \cos(E) \\
&- 15\, (1 - e^2)^{1/2}\, s1^3\, \cos(E) + 33\, e^2\, (1 - e^2)^{1/2}\, s1\, \cos(E) \\
&+ 12\, (1 - e^2)^{1/2}\, s1\, \cos(E))\, za2/(4\, (1 - e^2)^{1/2})
\end{aligned}$$

$$n\delta I_3 =$$

$$\begin{aligned}
&s3\, (30\, e^3\, s1\, s2\, \sin(4\,E)\, \sin(w) - 30\, e\, s1\, s2\, \sin(4\,E)\, \sin(w) \\
&- 15\, e^3\, (1 - e^2)^{1/2}\, s2^2\, \cos(4\,E)\, \sin(w) + 15\, e\, (1 - e^2)^{1/2}\, s2^2\, \cos(4\,E)\, \sin(w) \\
&- 15\, e\, (1 - e^2)^{1/2}\, s1^2\, \cos(4\,E)\, \sin(w) + 3\, e^3\, (1 - e^2)^{1/2}\, \cos(4\,E)\, \sin(w) \\
&- 80\, e^4\, s1\, s2\, \sin(3\,E)\, \sin(w) + 80\, s1\, s2\, \sin(3\,E)\, \sin(w) \\
&+ 40\, e^2\, (1 - e^2)^{1/2}\, s2^2\, \cos(3\,E)\, \sin(w) - 40\, (1 - e^2)^{1/2}\, s2^2\, \cos(3\,E)\, \sin(w)
\end{aligned}$$

```
      2        2 1/2   2                                     2 1/2   2
+ 80 e  (1 - e )    s1  cos(3 E) sin(w) + 40 (1 - e )    s1  cos(3 E) sin(w)
      2        2 1/2                       3
- 24 e  (1 - e )    cos(3 E) sin(w) + 240 e  s1 s2 sin(2 E) sin(w)
                                     3       2 1/2   2
- 240 e s1 s2 sin(2 E) sin(w) + 60 e  (1 - e )    s2  cos(2 E) sin(w)
             2 1/2   2
- 60 e (1 - e )    s2  cos(2 E) sin(w)
       3        2 1/2   2
- 120 e  (1 - e )    s1  cos(2 E) sin(w)
              2 1/2   2                      3       2 1/2
- 300 e (1 - e )    s1  cos(2 E) sin(w) + 12 e  (1 - e )    cos(2 E) sin(w)
             2 1/2                        4
+ 72 e (1 - e )    cos(2 E) sin(w) + 240 e  s1 s2 sin(E) sin(w)
                                   2       2 1/2   2
- 240 s1 s2 sin(E) sin(w) - 360 e  (1 - e )    s2  cos(E) sin(w)
            2 1/2   2                        2       2 1/2   2
+ 360 (1 - e )    s2  cos(E) sin(w) + 720 e  (1 - e )    s1  cos(E) sin(w)
            2 1/2   2                       2       2 1/2
+ 120 (1 - e )    s1  cos(E) sin(w) - 72 e  (1 - e )    cos(E) sin(w)
           2 1/2                 3   2
- 96 (1 - e )    cos(E) sin(w) - 15 e  s2  sin(4 E) cos(w)
          2                          2                         3
+ 15 e s2  sin(4 E) cos(w) - 15 e s1  sin(4 E) cos(w) + 3 e  sin(4 E) cos(w)
             2 1/2                        4   2
+ 30 e (1 - e )    s1 s2 cos(4 E) cos(w) + 40 e  s2  sin(3 E) cos(w)
       2                         2   2                          2
- 40 s2  sin(3 E) cos(w) + 120 e  s1  sin(3 E) cos(w) + 40 s1  sin(3 E) cos(w)
     4                        2
- 8 e  sin(3 E) cos(w) - 24 e  sin(3 E) cos(w)
       2        2 1/2
- 160 e  (1 - e )    s1 s2 cos(3 E) cos(w)
           2 1/2                          3   2
- 80 (1 - e )    s1 s2 cos(3 E) cos(w) - 120 e  s2  sin(2 E) cos(w)
           2                        3   2
+ 120 e s2  sin(2 E) cos(w) - 360 e  s1  sin(2 E) cos(w)
           2                       3
- 480 e s1  sin(2 E) cos(w) + 96 e  sin(2 E) cos(w) + 72 e sin(2 E) cos(w)
       3        2 1/2
+ 240 e  (1 - e )    s1 s2 cos(2 E) cos(w)
              2 1/2                         4   2
+ 600 e (1 - e )    s1 s2 cos(2 E) cos(w) - 120 e  s2  sin(E) cos(w)
        2                       4   2                      2   2
+ 120 s2  sin(E) cos(w) + 480 e  s1  sin(E) cos(w) + 2520 e  s1  sin(E) cos(w)
        2                     4                         2
+ 360 s1  sin(E) cos(w) - 72 e  sin(E) cos(w) - 504 e  sin(E) cos(w)
                                2       2 1/2
- 96 sin(E) cos(w) - 1440 e  (1 - e )    s1 s2 cos(E) cos(w)
            2 1/2                                        2 1/2
- 240 (1 - e )    s1 s2 cos(E) cos(w)) za2/(64 (1 - e )    )
```

$n\delta\Omega_3 =$

```
                 3   2                               2
      - s3 (15 e  s2  sin(4 E) sin(w) - 15 e s2  sin(4 E) sin(w)
          2                          3
+ 15 e s1  sin(4 E) sin(w) - 3 e  sin(4 E) sin(w)
```

$$- 30 e (1 - e^{2})^{1/2} s1\, s2 \cos(4 E) \sin(w) - 40 e^{4} s2^{2} \sin(3 E) \sin(w)$$
$$+ 40 s2^{2} \sin(3 E) \sin(w) - 120 e^{2} s1^{2} \sin(3 E) \sin(w) - 40 s1^{2} \sin(3 E) \sin(w)$$
$$+ 8 e^{4} \sin(3 E) \sin(w) + 24 e^{2} \sin(3 E) \sin(w)$$
$$+ 160 e^{2} (1 - e^{2})^{1/2} s1\, s2 \cos(3 E) \sin(w)$$
$$+ 80 (1 - e^{2})^{1/2} s1\, s2 \cos(3 E) \sin(w) + 120 e^{3} s2^{2} \sin(2 E) \sin(w)$$
$$- 120 e\, s2^{2} \sin(2 E) \sin(w) + 360 e^{3} s1^{2} \sin(2 E) \sin(w)$$
$$+ 480 e\, s1^{2} \sin(2 E) \sin(w) - 96 e^{3} \sin(2 E) \sin(w) - 72 e \sin(2 E) \sin(w)$$
$$- 240 e^{3} (1 - e^{2})^{1/2} s1\, s2 \cos(2 E) \sin(w)$$
$$- 600 e (1 - e^{2})^{1/2} s1\, s2 \cos(2 E) \sin(w) + 120 e^{4} s2^{2} \sin(E) \sin(w)$$
$$- 120 s2^{2} \sin(E) \sin(w) - 480 e^{4} s1^{2} \sin(E) \sin(w) - 2520 e^{2} s1^{2} \sin(E) \sin(w)$$
$$- 360 s1^{2} \sin(E) \sin(w) + 72 e^{4} \sin(E) \sin(w) + 504 e^{2} \sin(E) \sin(w)$$
$$+ 96 \sin(E) \sin(w) + 1440 e^{2} (1 - e^{2})^{1/2} s1\, s2 \cos(E) \sin(w)$$
$$+ 240 (1 - e^{2})^{1/2} s1\, s2 \cos(E) \sin(w) + 30 e^{3} s1\, s2 \sin(4 E) \cos(w)$$
$$- 30 e\, s1\, s2 \sin(4 E) \cos(w) - 15 e^{3} (1 - e^{2})^{1/2} s2^{2} \cos(4 E) \cos(w)$$
$$+ 15 e (1 - e^{2})^{1/2} s2^{2} \cos(4 E) \cos(w) - 15 e (1 - e^{2})^{1/2} s1^{2} \cos(4 E) \cos(w)$$
$$+ 3 e^{3} (1 - e^{2})^{1/2} \cos(4 E) \cos(w) - 80 e^{4} s1\, s2 \sin(3 E) \cos(w)$$
$$+ 80 s1\, s2 \sin(3 E) \cos(w) + 40 e^{2} (1 - e^{2})^{1/2} s2^{2} \cos(3 E) \cos(w)$$
$$- 40 (1 - e^{2})^{1/2} s2^{2} \cos(3 E) \cos(w) + 80 e^{2} (1 - e^{2})^{1/2} s1^{2} \cos(3 E) \cos(w)$$
$$+ 40 (1 - e^{2})^{1/2} s1^{2} \cos(3 E) \cos(w) - 24 e^{2} (1 - e^{2})^{1/2} \cos(3 E) \cos(w)$$
$$+ 240 e^{3} s1\, s2 \sin(2 E) \cos(w) - 240 e\, s1\, s2 \sin(2 E) \cos(w)$$
$$+ 60 e^{3} (1 - e^{2})^{1/2} s2^{2} \cos(2 E) \cos(w) - 60 e (1 - e^{2})^{1/2} s2^{2} \cos(2 E) \cos(w)$$
$$- 120 e^{3} (1 - e^{2})^{1/2} s1^{2} \cos(2 E) \cos(w)$$
$$- 300 e (1 - e^{2})^{1/2} s1^{2} \cos(2 E) \cos(w) + 12 e^{3} (1 - e^{2})^{1/2} \cos(2 E) \cos(w)$$
$$+ 72 e (1 - e^{2})^{1/2} \cos(2 E) \cos(w) + 240 e^{4} s1\, s2 \sin(E) \cos(w)$$
$$- 240 s1\, s2 \sin(E) \cos(w) - 360 e^{2} (1 - e^{2})^{1/2} s2^{2} \cos(E) \cos(w)$$
$$+ 360 (1 - e^{2})^{1/2} s2^{2} \cos(E) \cos(w) + 720 e^{2} (1 - e^{2})^{1/2} s1^{2} \cos(E) \cos(w)$$
$$+ 120 (1 - e^{2})^{1/2} s1^{2} \cos(E) \cos(w) - 72 e^{2} (1 - e^{2})^{1/2} \cos(E) \cos(w)$$
$$- 96 (1 - e^{2})^{1/2} \cos(E) \cos(w)) za2/(64 (1 - e^{2})^{1/2} \sin(i))$$

$$n\delta\epsilon_3 = \frac{1-\epsilon^2}{2a\epsilon} n\delta a_3 +$$

```
           2 1/2      3   3                    3
      (1 - e )    (15 e  s2  sin(4 E) - 15 e s2  sin(4 E)
      3   2                         2                   3
- 30 e  s1  s2 sin(4 E) + 45 e s1  s2 sin(4 E) - 3 e  s2 sin(4 E)
      3       2 1/2      2                        2 1/2      2
+ 15 e  (1 - e )    s1 s2  cos(4 E) - 45 e (1 - e )    s1 s2  cos(4 E)
             2 1/2   3                3       2 1/2
+ 15 e (1 - e )    s1  cos(4 E) - 3 e  (1 - e )    s1 cos(4 E)
      4   3                3                4   2
- 40 e  s2  sin(3 E) + 40 s2  sin(3 E) + 80 e  s1  s2 sin(3 E)
       2   2                         2                   4
- 120 e  s1  s2 sin(3 E) - 120 s1  s2 sin(3 E) + 8 e  s2 sin(3 E)
      2                    2       2 1/2      2
+ 24 e  s2 sin(3 E) + 120 e  (1 - e )    s1 s2  cos(3 E)
            2 1/2      2                2       2 1/2   3
+ 120 (1 - e )    s1 s2  cos(3 E) - 80 e  (1 - e )    s1  cos(3 E)
           2 1/2   3                  2       2 1/2
- 40 (1 - e )    s1  cos(3 E) + 24 e  (1 - e )    s1 cos(3 E)
       3   3                   3                3   2
+ 120 e  s2  sin(2 E) - 120 e s2  sin(2 E) + 120 e  s1  s2 sin(2 E)
          2                 3
+ 720 e s1  s2 sin(2 E) - 96 e  s2 sin(2 E) - 72 e s2 sin(2 E)
       3       2 1/2      2                         2 1/2      2
- 300 e  (1 - e )    s1 s2  cos(2 E) - 540 e (1 - e )    s1 s2  cos(2 E)
       3       2 1/2   3                         2 1/2   3
+ 120 e  (1 - e )    s1  cos(2 E) + 300 e (1 - e )    s1  cos(2 E)
      3       2 1/2                          2 1/2
- 12 e  (1 - e )    s1 cos(2 E) - 72 e (1 - e )    s1 cos(2 E)
       4   3                  3              4   2
+ 120 e  s2  sin(E) - 120 s2  sin(E) - 720 e  s1  s2 sin(E)
        2   2                  2                 4
- 2520 e  s1  s2 sin(E) - 120 s1  s2 sin(E) + 72 e  s2 sin(E)
       2                                      2       2 1/2      2
+ 504 e  s2 sin(E) + 96 s2 sin(E) + 1800 e  (1 - e )    s1 s2  cos(E)
            2 1/2      2               2       2 1/2   3
- 120 (1 - e )    s1 s2  cos(E) - 720 e  (1 - e )    s1  cos(E)
            2 1/2   3              2       2 1/2
- 120 (1 - e )    s1  cos(E) + 72 e  (1 - e )    s1 cos(E)
           2 1/2
+ 96 (1 - e )    s1 cos(E)) za2/(64 e)
```

$$n\delta\omega_3 = -\cos(I) n\delta\Omega_3 +$$

```
           4       2              2      2                       2
      - (15 e  s1 s2  sin(4 E) - 60 e  s1 s2  sin(4 E) + 45 s1 s2  sin(4 E)
      2   3                3               4                  2
+ 15 e  s1  sin(4 E) - 15 s1  sin(4 E) - 3 e  s1 sin(4 E) + 3 e  s1 sin(4 E)
      2       2 1/2   3                      2 1/2   3
+ 15 e  (1 - e )    s2  cos(4 E) - 15 (1 - e )    s2  cos(4 E)
      2       2 1/2   2                          2 1/2   2
- 30 e  (1 - e )    s1  s2 cos(4 E) + 45 (1 - e )    s1  s2 cos(4 E)
     2       2 1/2                   5       2               3       2
- 3 e  (1 - e )    s2 cos(4 E) + 40 e  s1 s2  sin(3 E) + 80 e  s1 s2  sin(3 E)
```

```
                 2                  3   3                      3
 - 120 e s1 s2  sin(3 E) - 40 e  s1  sin(3 E) + 40 e s1  sin(3 E)
      5                 3                 3        2 1/2   3
 - 8 e  s1 sin(3 E) + 8 e  s1 sin(3 E) - 40 e  (1 - e )    s2  cos(3 E)
                 2 1/2   3                    2 1/2   2
 + 40 e (1 - e )     s2  cos(3 E) - 120 e (1 - e )    s1  s2 cos(3 E)
       3      2 1/2                       4       2
 + 24 e  (1 - e )     s2 cos(3 E) - 480 e  s1 s2  sin(2 E)
        2       2                       2                 4   3
 + 720 e  s1 s2  sin(2 E) - 240 s1 s2  sin(2 E) - 120 e  s1  sin(2 E)
         3                4                       2
 + 120 s1  sin(2 E) + 168 e  s1 sin(2 E) - 144 e  s1 sin(2 E) - 24 s1 sin(2 E)
       2       2 1/2   3                    2 1/2   3
 - 60 e  (1 - e )     s2  cos(2 E) + 60 (1 - e )    s2  cos(2 E)
        4        2 1/2   2                   2        2 1/2   2
 + 240 e  (1 - e )     s1  s2 cos(2 E) + 480 e  (1 - e )    s1  s2 cos(2 E)
                2 1/2   2                   4       2 1/2
 - 300 (1 - e )     s1  s2 cos(2 E) - 48 e  (1 - e )    s2 cos(2 E)
       2       2 1/2                      2 1/2
 - 60 e  (1 - e )'  s2 cos(2 E) + 24 (1 - e )    s2 cos(2 E)
        5       2              3      2                       2
 - 120 e  s1 s2  sin(E) - 240 e  s1 s2  sin(E) + 360 e s1 s2  sin(E)
        5   3               3   3                      3
 + 480 e  s1  sin(E) + 1080 e  s1  sin(E) - 1560 e s1  sin(E)
        5                    3
 - 264 e  s1 sin(E) - 600 e  s1 sin(E) + 864 e s1 sin(E)
        3        2 1/2   3                    2 1/2   3
 + 360 e  (1 - e )     s2  cos(E) - 360 e (1 - e )    s2  cos(E)
         3        2 1/2   2                          2 1/2   2
 - 2400 e  (1 - e )     s1  s2 cos(E) + 1560 e (1 - e )    s1  s2 cos(E)
        3        2 1/2                         2 1/2
 + 264 e  (1 - e )     s2 cos(E) - 96 e (1 - e )    s2 cos(E)) za2
                2 1/2
/(64 e (1 - e )    )
```

$$n\delta M_3 = -\sqrt{1-\epsilon^2}\,[n\delta\omega_3 + \cos(I)n\delta\Omega_3] +$$

```
              3        2 1/2              3               2 1/2             3
     3 (- 15 e  (1 - e )     cos(4 E) s2  + 15 e (1 - e )     cos(4 E) s2
       2       2 1/2               3               2 1/2            3
 + 40 e  (1 - e )     cos(3 E) s2  - 40 (1 - e )     cos(3 E) s2
       3       2 1/2              3               2 1/2            3
 + 60 e  (1 - e )     cos(2 E) s2  - 60 e (1 - e )     cos(2 E) s2
        2       2 1/2            3                2 1/2          3
 - 360 e  (1 - e )     cos(E) s2  + 360 (1 - e )     cos(E) s2
       3                 2                 2       4                 2
 + 45 e  sin(4 E) s1 s2  - 45 e sin(4 E) s1 s2  - 120 e  sin(3 E) s1 s2
                        2        3                 2                     2
 + 120 sin(3 E) s1 s2  + 360 e  sin(2 E) s1 s2  - 360 e sin(2 E) s1 s2
        4               2        2               2                   2
 + 120 e  sin(E) s1 s2  + 240 e  sin(E) s1 s2  - 360 sin(E) s1 s2
                2 1/2                 2           2       2 1/2               2
 - 45 e (1 - e )     cos(4 E) s1  s2 + 240 e  (1 - e )     cos(3 E) s1  s2
                2 1/2             2            3       2 1/2               2
 + 120 (1 - e )     cos(3 E) s1  s2 - 360 e  (1 - e )     cos(2 E) s1  s2
```

```
              2 1/2            2            2        2 1/2         2
- 900 e (1 - e )    cos(2 E) s1  s2 + 2160 e   (1 - e )   cos(E) s1  s2
            2 1/2          2         3       2 1/2
+ 360 (1 - e )    cos(E) s1  s2 + 9 e  (1 - e )    cos(4 E) s2
      2       2 1/2                   3       2 1/2
- 72 e  (1 - e )    cos(3 E) s2 + 36 e  (1 - e )    cos(2 E) s2
              2 1/2                    2       2 1/2
+ 216 e (1 - e )    cos(2 E) s2 - 216 e  (1 - e )    cos(E) s2
            2 1/2                             3        2            3
- 288 (1 - e )    cos(E) s2 + 15 e sin(4 E) s1  - 120 e  sin(3 E) s1
                3        3            3                    3
- 40 sin(3 E) s1  + 360 e  sin(2 E) s1  + 480 e sin(2 E) s1
       4          3         2          3                3      3
- 320 e  sin(E) s1  - 2280 e  sin(E) s1  - 360 sin(E) s1  - 9 e  sin(4 E) s1
      4                   2                    3
+ 24 e  sin(3 E) s1 + 72 e  sin(3 E) s1 - 288 e  sin(2 E) s1
                           4                   2
- 216 e sin(2 E) s1 + 168 e  sin(E) s1 + 1320 e  sin(E) s1 + 288 sin(E) s1)

za2/64
```

In order to express the long period averaging of the above over the mean anomaly of the Moon, it suffices to average various products of $S1, S2$ and $S3$ with za for the $l = 2$ harmonic and with $za2$ for the $l = 3$ harmonic. For example, if one would like to propagate the eccentricity as it is influenced by the $l = 2$ harmonic, one uses the expressions for $\langle \dot{e} \rangle_2$ and δe_2 above along with the expressions for $\langle zaS1S2 \rangle$ and $\delta(zaS1S2)$ below. After these are evaluated, the doubly averaged secular rate is computed

$$\langle\langle \dot{\epsilon} \rangle\rangle_2 = \frac{-15\epsilon\sqrt{1-\epsilon^2}}{2} \langle zaS1S2 \rangle$$

and e is propagated by the formula

$$\epsilon(t) = \epsilon_0 + \langle\langle \dot{\epsilon} \rangle\rangle_2 (t - t_0) - \frac{15\epsilon\sqrt{1-\epsilon^2}}{2} \delta(zaS1S2) + \delta\epsilon_2,$$

where $\langle\langle \dot{e} \rangle\rangle_2$ is a function of the initial conditions, the long period correction is a function of the updated mean elements, and δe_2 is a function of the updated mean elements with long period corrections.

The presentation of the expressions below requires the definition of two quantities: for the $l = 2$ harmonic we define

$$zb \equiv \left(\frac{Gm_k}{\mu} \right) \left(\frac{a}{a_k} \right)^3 \frac{1}{(1 - \epsilon_k^2)^{3/2}},$$

and for the $l = 3$ harmonic we define

$$zb2 \equiv zb\left(\frac{a}{a_k}\right).$$

The result of averaging the pertinent products for the rates from $l = 2$ harmonic over the period of the Moon is

$$\langle za \rangle = nzb$$

$$\langle zaS1^2 \rangle =$$

```
          2         2           2
  (sin (apk) cos (jk) + cos (apk)) n zb
  -------------------------------------
                    2
```

$$\langle zaS2^2 \rangle =$$

```
          2         2           2
  (cos (apk) cos (jk) + sin (apk)) n zb
  -------------------------------------
                    2
```

$$\langle zaS1S2 \rangle =$$

```
                                2
    cos(apk) sin(apk) sin (jk) n zb
  - -------------------------------
                   2
```

$$\langle zaS1S3 \rangle =$$

```
    sin(apk) cos(jk) sin(jk) n zb
  - -----------------------------
                  2
```

$$\langle zaS2S3 \rangle =$$

```
    cos(apk) cos(jk) sin(jk) n zb
  - -----------------------------
                  2
```

with long period corrections

$\delta(za) =$

```
ek sin(fk) n zb
---------------
      nk
```

$\delta(zaS1^2) =$

```
           2                      2
     n (cos (apk) ek cos(3 fk) cos (jk) cos(wk) sin(wk)
                  2
- ek cos(3 fk) cos (jk) cos(wk) sin(wk)
       2                   2
+ 3 cos (apk) cos(2 fk) cos (jk) cos(wk) sin(wk)
                 2
- 3 cos(2 fk) cos (jk) cos(wk) sin(wk)
       2                    2
+ 3 cos (apk) ek cos(fk) cos (jk) cos(wk) sin(wk)
                  2
- 3 ek cos(fk) cos (jk) cos(wk) sin(wk)

+ 4 cos(apk) sin(apk) ek sin(fk) cos(2 fk) cos(jk) cos(wk) sin(wk)

+ 12 cos(apk) sin(apk) cos(fk) sin(fk) cos(jk) cos(wk) sin(wk)

+ 8 cos(apk) sin(apk) ek sin(fk) cos(jk) cos(wk) sin(wk)
     2
+ cos (apk) ek cos(3 fk) cos(wk) sin(wk)
       2
+ 3 cos (apk) cos(2 fk) cos(wk) sin(wk)
       2
+ 3 cos (apk) ek cos(fk) cos(wk) sin(wk)
       2                              2        2
+ 2 cos (apk) ek sin(fk) cos(2 fk) cos (jk) cos (wk)
                            2        2
- 2 ek sin(fk) cos(2 fk) cos (jk) cos (wk)
       2                         2        2
+ 6 cos (apk) cos(fk) sin(fk) cos (jk) cos (wk)
                       2        2            2                    2
- 6 cos(fk) sin(fk) cos (jk) cos (wk) + 4 cos (apk) ek sin(fk) cos (jk)
   2                       2        2
cos (wk) - 4 ek sin(fk) cos (jk) cos (wk)
                                              2
- 2 cos(apk) sin(apk) ek cos(3 fk) cos(jk) cos (wk)
                                           2
- 6 cos(apk) sin(apk) cos(2 fk) cos(jk) cos (wk)
                                            2
- 6 cos(apk) sin(apk) ek cos(fk) cos(jk) cos (wk)
       2                              2
+ 2 cos (apk) ek sin(fk) cos(2 fk) cos (wk)
       2                         2            2                    2
+ 6 cos (apk) cos(fk) sin(fk) cos (wk) + 4 cos (apk) ek sin(fk) cos (wk)
     2                              2                               2
- cos (apk) ek sin(fk) cos(2 fk) cos (jk) + ek sin(fk) cos(2 fk) cos (jk)
       2                         2                            2
- 3 cos (apk) cos(fk) sin(fk) cos (jk) + 3 cos(fk) sin(fk) cos (jk)
```

$$- 5 \cos^2(apk)\ ek \sin(fk) \cos^2(jk) + 5\ ek \sin(fk) \cos^2(jk)$$

$$+ \cos(apk) \sin(apk)\ ek \cos(3\ fk) \cos(jk)$$

$$+ 3 \cos(apk) \sin(apk) \cos(2\ fk) \cos(jk)$$

$$+ 3 \cos(apk) \sin(apk)\ ek \cos(fk) \cos(jk) - \cos^2(apk)\ ek \sin(fk) \cos(2\ fk)$$

$$- 3 \cos^2(apk) \cos(fk) \sin(fk) + \cos^2(apk)\ ek \sin(fk))\ zb/(6\ nk)$$

$$\delta(zaS2^2) =$$

$$- n\ (\cos^2(apk)\ ek \cos(3\ fk) \cos^2(jk) \cos(wk) \sin(wk)$$

$$+ 3 \cos^2(apk) \cos(2\ fk) \cos^2(jk) \cos(wk) \sin(wk)$$

$$+ 3 \cos^2(apk)\ ek \cos(fk) \cos^2(jk) \cos(wk) \sin(wk)$$

$$+ 4 \cos(apk) \sin(apk)\ ek \sin(fk) \cos(2\ fk) \cos(jk) \cos(wk) \sin(wk)$$

$$+ 12 \cos(apk) \sin(apk) \cos(fk) \sin(fk) \cos(jk) \cos(wk) \sin(wk)$$

$$+ 8 \cos(apk) \sin(apk)\ ek \sin(fk) \cos(jk) \cos(wk) \sin(wk)$$

$$+ \cos^2(apk)\ ek \cos(3\ fk) \cos(wk) \sin(wk) - ek \cos(3\ fk) \cos(wk) \sin(wk)$$

$$+ 3 \cos^2(apk) \cos(2\ fk) \cos(wk) \sin(wk) - 3 \cos(2\ fk) \cos(wk) \sin(wk)$$

$$+ 3 \cos^2(apk)\ ek \cos(fk) \cos(wk) \sin(wk) - 3\ ek \cos(fk) \cos(wk) \sin(wk)$$

$$+ 2 \cos^2(apk)\ ek \sin(fk) \cos(2\ fk) \cos^2(jk) \cos^2(wk)$$

$$+ 6 \cos^2(apk) \cos(fk) \sin(fk) \cos^2(jk) \cos^2(wk)$$

$$+ 4 \cos^2(apk)\ ek \sin(fk) \cos^2(jk) \cos^2(wk)$$

$$- 2 \cos(apk) \sin(apk)\ ek \cos(3\ fk) \cos(jk) \cos^2(wk)$$

$$- 6 \cos(apk) \sin(apk) \cos(2\ fk) \cos(jk) \cos^2(wk)$$

$$- 6 \cos(apk) \sin(apk)\ ek \cos(fk) \cos(jk) \cos^2(wk)$$

$$+ 2 \cos^2(apk)\ ek \sin(fk) \cos(2\ fk) \cos^2(wk) - 2\ ek \sin(fk) \cos(2\ fk) \cos^2(wk)$$

$$+ 6 \cos^2(apk) \cos(fk) \sin(fk) \cos^2(wk) - 6 \cos(fk) \sin(fk) \cos^2(wk)$$

$$+ 4 \cos^2(apk)\ ek \sin(fk) \cos^2(wk) - 4\ ek \sin(fk) \cos^2(wk)$$

$$- \cos^2(apk)\ ek \sin(fk) \cos(2\ fk) \cos^2(jk)$$

$$- 3 \cos^2(apk) \cos(fk) \sin(fk) \cos^2(jk) - 5 \cos^2(apk)\ ek \sin(fk) \cos^2(jk)$$

$$+ \cos(apk) \sin(apk)\ ek \cos(3\ fk) \cos(jk)$$

$$+ 3 \cos(apk) \sin(apk) \cos(2\ fk) \cos(jk)$$

```
                                              2
+ 3 cos(apk) sin(apk) ek cos(fk) cos(jk) - cos (apk) ek sin(fk) cos(2 fk)
                              2
+ ek sin(fk) cos(2 fk) - 3 cos (apk) cos(fk) sin(fk) + 3 cos(fk) sin(fk)
     2
+ cos (apk) ek sin(fk) - ek sin(fk)) zb/(6 nk)
```

$\delta(zaS1S2) =$

```
                                              2
     - n (2 cos(apk) sin(apk) ek cos(3 fk) cos (jk) cos(wk) sin(wk)
                                   2
+ 6 cos(apk) sin(apk) cos(2 fk) cos (jk) cos(wk) sin(wk)
                                    2
+ 6 cos(apk) sin(apk) ek cos(fk) cos (jk) cos(wk) sin(wk)
       2
- 8 cos (apk) ek sin(fk) cos(2 fk) cos(jk) cos(wk) sin(wk)

+ 4 ek sin(fk) cos(2 fk) cos(jk) cos(wk) sin(wk)
        2
- 24 cos (apk) cos(fk) sin(fk) cos(jk) cos(wk) sin(wk)

+ 12 cos(fk) sin(fk) cos(jk) cos(wk) sin(wk)
        2
- 16 cos (apk) ek sin(fk) cos(jk) cos(wk) sin(wk)

+ 8 ek sin(fk) cos(jk) cos(wk) sin(wk)

+ 2 cos(apk) sin(apk) ek cos(3 fk) cos(wk) sin(wk)

+ 6 cos(apk) sin(apk) cos(2 fk) cos(wk) sin(wk)

+ 6 cos(apk) sin(apk) ek cos(fk) cos(wk) sin(wk)
                                              2        2
+ 4 cos(apk) sin(apk) ek sin(fk) cos(2 fk) cos (jk) cos (wk)
                                          2        2
+ 12 cos(apk) sin(apk) cos(fk) sin(fk) cos (jk) cos (wk)
                                    2        2
+ 8 cos(apk) sin(apk) ek sin(fk) cos (jk) cos (wk)
       2                              2                                  2
+ 4 cos (apk) ek cos(3 fk) cos(jk) cos (wk) - 2 ek cos(3 fk) cos(jk) cos (wk)
        2                           2                              2
+ 12 cos (apk) cos(2 fk) cos(jk) cos (wk) - 6 cos(2 fk) cos(jk) cos (wk)
        2                            2                               2
+ 12 cos (apk) ek cos(fk) cos(jk) cos (wk) - 6 ek cos(fk) cos(jk) cos (wk)
                                              2
+ 4 cos(apk) sin(apk) ek sin(fk) cos(2 fk) cos (wk)
                                          2
+ 12 cos(apk) sin(apk) cos(fk) sin(fk) cos (wk)
                                    2
+ 8 cos(apk) sin(apk) ek sin(fk) cos (wk)
                                              2
- 2 cos(apk) sin(apk) ek sin(fk) cos(2 fk) cos (jk)
                                         2
- 6 cos(apk) sin(apk) cos(fk) sin(fk) cos (jk)
                                     2            2
- 10 cos(apk) sin(apk) ek sin(fk) cos (jk) - 2 cos (apk) ek cos(3 fk) cos(jk)
```

```
                              2
 + ek cos(3 fk) cos(jk) - 6 cos (apk) cos(2 fk) cos(jk) + 3 cos(2 fk) cos(jk)
          2
 - 6 cos (apk) ek cos(fk) cos(jk) + 3 ek cos(fk) cos(jk)

 - 2 cos(apk) sin(apk) ek sin(fk) cos(2 fk)

 - 6 cos(apk) sin(apk) cos(fk) sin(fk) + 2 cos(apk) sin(apk) ek sin(fk)) zb

/(12 nk)
```

$\delta(zaS1S3) =$

```
   sin(jk) n (2 sin(apk) ek cos(3 fk) cos(jk) cos(wk) sin(wk)

 + 6 sin(apk) cos(2 fk) cos(jk) cos(wk) sin(wk)

 + 6 sin(apk) ek cos(fk) cos(jk) cos(wk) sin(wk)

 - 4 cos(apk) ek sin(fk) cos(2 fk) cos(wk) sin(wk)

 - 12 cos(apk) cos(fk) sin(fk) cos(wk) sin(wk)

 - 8 cos(apk) ek sin(fk) cos(wk) sin(wk)
                                                   2
 + 4 sin(apk) ek sin(fk) cos(2 fk) cos(jk) cos (wk)
                                               2
 + 12 sin(apk) cos(fk) sin(fk) cos(jk) cos (wk)
                                           2                                 2
 + 8 sin(apk) ek sin(fk) cos(jk) cos (wk) + 2 cos(apk) ek cos(3 fk) cos (wk)
                                2                                   2
 + 6 cos(apk) cos(2 fk) cos (wk) + 6 cos(apk) ek cos(fk) cos (wk)

 - 2 sin(apk) ek sin(fk) cos(2 fk) cos(jk) - 6 sin(apk) cos(fk) sin(fk) cos(jk)

 - 10 sin(apk) ek sin(fk) cos(jk) - cos(apk) ek cos(3 fk)

 - 3 cos(apk) cos(2 fk) - 3 cos(apk) ek cos(fk)) zb/(12 nk)
```

$\delta(zaS2S3) =$

```
   sin(jk) n (2 cos(apk) ek cos(3 fk) cos(jk) cos(wk) sin(wk)

 + 6 cos(apk) cos(2 fk) cos(jk) cos(wk) sin(wk)

 + 6 cos(apk) ek cos(fk) cos(jk) cos(wk) sin(wk)

 + 4 sin(apk) ek sin(fk) cos(2 fk) cos(wk) sin(wk)

 + 12 sin(apk) cos(fk) sin(fk) cos(wk) sin(wk)

 + 8 sin(apk) ek sin(fk) cos(wk) sin(wk)
                                                   2
 + 4 cos(apk) ek sin(fk) cos(2 fk) cos(jk) cos (wk)
```

```
                                               2
+ 12 cos(apk) cos(fk) sin(fk) cos(jk) cos (wk)
                                          2                                    2
+ 8 cos(apk) ek sin(fk) cos(jk) cos (wk) - 2 sin(apk) ek cos(3 fk) cos (wk)
                                  2                              2
- 6 sin(apk) cos(2 fk) cos (wk) - 6 sin(apk) ek cos(fk) cos (wk)

- 2 cos(apk) ek sin(fk) cos(2 fk) cos(jk) - 6 cos(apk) cos(fk) sin(fk) cos(jk)

- 10 cos(apk) ek sin(fk) cos(jk) + sin(apk) ek cos(3 fk)

+ 3 sin(apk) cos(2 fk) + 3 sin(apk) ek cos(fk)) zb/(12 nk)
```

The result of averaging the pertinent products for the rates from the $l = 3$ harmonic over the period of the Moon is

$\langle za2S1 \rangle =$

```
- ek n (sin(apk) cos(jk) sin(wk) + cos(apk) cos(wk)) zb2
```

$\langle za2S2 \rangle =$

```
- ek n (cos(apk) cos(jk) sin(wk) - sin(apk) cos(wk)) zb2
```

$\langle za2S3 \rangle =$

```
ek sin(jk) n sin(wk) zb2
```

$\langle za2S1^3 \rangle =$

```
         2          2          2          2
3 ek (cos (apk) cos (jk) - cos (jk) - cos (apk)) n

     (sin(apk) cos(jk) sin(wk) + cos(apk) cos(wk)) zb2/4
```

$\langle za2S2^3 \rangle =$

```
           2          2          2
- 3 ek (cos (apk) cos (jk) - cos (apk) + 1) n

       (cos(apk) cos(jk) sin(wk) - sin(apk) cos(wk)) zb2/4
```

$$\langle za2S1S2^2 \rangle =$$

$$\begin{aligned}
& - \mathrm{ek}\; n \,(3 \cos^2(\mathrm{apk}) \sin(\mathrm{apk}) \cos^3(\mathrm{jk}) \sin(\mathrm{wk}) \\
& - 3 \cos^2(\mathrm{apk}) \sin(\mathrm{apk}) \cos(\mathrm{jk}) \sin(\mathrm{wk}) + \sin(\mathrm{apk}) \cos(\mathrm{jk}) \sin(\mathrm{wk}) \\
& + 3 \cos^3(\mathrm{apk}) \cos^2(\mathrm{jk}) \cos(\mathrm{wk}) - 2 \cos(\mathrm{apk}) \cos^2(\mathrm{jk}) \cos(\mathrm{wk}) \\
& - 3 \cos^3(\mathrm{apk}) \cos(\mathrm{wk}) + 3 \cos(\mathrm{apk}) \cos(\mathrm{wk}))\, \mathrm{zb2}/4
\end{aligned}$$

$$\langle za2S2S1^2 \rangle =$$

$$\begin{aligned}
& \mathrm{ek}\; n \,(3 \cos^3(\mathrm{apk}) \cos^3(\mathrm{jk}) \sin(\mathrm{wk}) - 3 \cos(\mathrm{apk}) \cos^3(\mathrm{jk}) \sin(\mathrm{wk}) \\
& - 3 \cos^3(\mathrm{apk}) \cos(\mathrm{jk}) \sin(\mathrm{wk}) + 2 \cos(\mathrm{apk}) \cos(\mathrm{jk}) \sin(\mathrm{wk}) \\
& - 3 \cos^2(\mathrm{apk}) \sin(\mathrm{apk}) \cos^2(\mathrm{jk}) \cos(\mathrm{wk}) + \sin(\mathrm{apk}) \cos^2(\mathrm{jk}) \cos(\mathrm{wk}) \\
& + 3 \cos^2(\mathrm{apk}) \sin(\mathrm{apk}) \cos(\mathrm{wk}))\, \mathrm{zb2}/4
\end{aligned}$$

$$\langle za2S1S2S3 \rangle =$$

$$\begin{aligned}
& \mathrm{ek} \sin(\mathrm{jk})\; n \,(3 \cos(\mathrm{apk}) \sin(\mathrm{apk}) \cos^2(\mathrm{jk}) \sin(\mathrm{wk}) \\
& - \cos(\mathrm{apk}) \sin(\mathrm{apk}) \sin(\mathrm{wk}) + 2 \cos^2(\mathrm{apk}) \cos(\mathrm{jk}) \cos(\mathrm{wk}) - \cos(\mathrm{jk}) \cos(\mathrm{wk})) \\
& \mathrm{zb2}/4
\end{aligned}$$

$$\langle za2S3S2^2 \rangle =$$

$$\begin{aligned}
& \mathrm{ek} \sin(\mathrm{jk})\; n \,(3 \cos^2(\mathrm{apk}) \cos^2(\mathrm{jk}) \sin(\mathrm{wk}) - \cos^2(\mathrm{apk}) \sin(\mathrm{wk}) + \sin(\mathrm{wk}) \\
& - 2 \cos(\mathrm{apk}) \sin(\mathrm{apk}) \cos(\mathrm{jk}) \cos(\mathrm{wk}))\, \mathrm{zb2}/4
\end{aligned}$$

$$\langle za2S3S1^2 \rangle =$$

$$\begin{aligned}
& - \mathrm{ek} \sin(\mathrm{jk})\; n \,(3 \cos^2(\mathrm{apk}) \cos^2(\mathrm{jk}) \sin(\mathrm{wk}) - 3 \cos^2(\mathrm{jk}) \sin(\mathrm{wk}) \\
& - \cos^2(\mathrm{apk}) \sin(\mathrm{wk}) - 2 \cos(\mathrm{apk}) \sin(\mathrm{apk}) \cos(\mathrm{jk}) \cos(\mathrm{wk}))\, \mathrm{zb2}/4
\end{aligned}$$

with long period corrections

$\delta(za2S1) =$

```
                       2
  - n (sin(apk) ek  sin(3 fk) cos(jk) sin(wk)

+ 6 sin(apk) ek sin(2 fk) cos(jk) sin(wk)
                  2
+ 9 sin(apk) ek  sin(fk) cos(jk) sin(wk) + 12 sin(apk) sin(fk) cos(jk) sin(wk)
                2
+ cos(apk) ek  cos(3 fk) sin(wk) + 6 cos(apk) ek cos(2 fk) sin(wk)
                  2
+ 3 cos(apk) ek  cos(fk) sin(wk) + 12 cos(apk) cos(fk) sin(wk)
                2
- sin(apk) ek  cos(3 fk) cos(jk) cos(wk)

- 6 sin(apk) ek cos(2 fk) cos(jk) cos(wk)
                  2
- 3 sin(apk) ek  cos(fk) cos(jk) cos(wk) - 12 sin(apk) cos(fk) cos(jk) cos(wk)
                2
+ cos(apk) ek  sin(3 fk) cos(wk) + 6 cos(apk) ek sin(2 fk) cos(wk)
                  2
+ 9 cos(apk) ek  sin(fk) cos(wk) + 12 cos(apk) sin(fk) cos(wk)) zb2/(12 nk)
```

$\delta(za2S2) =$

```
                       2
  - n (cos(apk) ek  sin(3 fk) cos(jk) sin(wk)

+ 6 cos(apk) ek sin(2 fk) cos(jk) sin(wk)
                  2
+ 9 cos(apk) ek  sin(fk) cos(jk) sin(wk) + 12 cos(apk) sin(fk) cos(jk) sin(wk)
                2
- sin(apk) ek  cos(3 fk) sin(wk) - 6 sin(apk) ek cos(2 fk) sin(wk)
                  2
- 3 sin(apk) ek  cos(fk) sin(wk) - 12 sin(apk) cos(fk) sin(wk)
                2
- cos(apk) ek  cos(3 fk) cos(jk) cos(wk)

- 6 cos(apk) ek cos(2 fk) cos(jk) cos(wk)
                  2
- 3 cos(apk) ek  cos(fk) cos(jk) cos(wk) - 12 cos(apk) cos(fk) cos(jk) cos(wk)
                2
- sin(apk) ek  sin(3 fk) cos(wk) - 6 sin(apk) ek sin(2 fk) cos(wk)
                  2
- 9 sin(apk) ek  sin(fk) cos(wk) - 12 sin(apk) sin(fk) cos(wk)) zb2/(12 nk)
```

$\delta(za2S3) =$

```
                    2
  sin(jk) n (ek  sin(3 fk) sin(wk) + 6 ek sin(2 fk) sin(wk)
       2                                         2
+ 9 ek  sin(fk) sin(wk) + 12 sin(fk) sin(wk) - ek  cos(3 fk) cos(wk)
                                  2
- 6 ek cos(2 fk) cos(wk) - 3 ek  cos(fk) cos(wk) - 12 cos(fk) cos(wk)) zb2

/(12 nk)
```

$\delta(za2S1^3) =$

$$
\begin{aligned}
&- n \,(24 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^3(jk) \cos^2(wk) \\
&\sin(wk) - 24 \sin(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&+ 120 \cos^2(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&- 120 \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&+ 104 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&- 104 \sin(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&+ 160 \cos^2(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&- 160 \sin(apk) \sin(fk) \cos(2\, fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&+ 360 \cos^2(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&- 360 \sin(apk)\, ek \cos(fk) \sin(fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&+ 112 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&- 112 \sin(apk)\, ek^2 \sin(fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&+ 80 \cos^2(apk) \sin(apk) \sin(fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&- 80 \sin(apk) \sin(fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&+ 36 \cos^3(apk)\, ek^2 \cos(5\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&- 36 \cos(apk)\, ek^2 \cos(5\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 180 \cos^3(apk)\, ek \cos(4\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&- 180 \cos(apk)\, ek \cos(4\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 120 \cos^3(apk)\, ek^2 \cos(3\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&- 120 \cos(apk)\, ek^2 \cos(3\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 240 \cos^3(apk) \cos(3\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&- 240 \cos(apk) \cos(3\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 360 \cos^3(apk)\, ek \cos(2\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&- 360 \cos(apk)\, ek \cos(2\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 180 \cos^3(apk)\, ek^2 \cos(fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&- 180 \cos(apk)\, ek^2 \cos(fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 72 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos(jk) \cos^2(wk) \sin(wk) \\
&+ 360 \cos^2(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \cos^2(wk) \sin(wk)
\end{aligned}
$$

$$+ 312 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$+ 480 \cos^2(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$+ 1080 \cos^2(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$+ 336 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$+ 240 \cos^2(apk) \sin(apk) \sin(fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$+ 12 \cos^3(apk)\, ek^2 \cos(5\, fk) \cos^2(wk) \sin(wk)$$
$$+ 60 \cos^3(apk)\, ek \cos(4\, fk) \cos^2(wk) \sin(wk)$$
$$+ 40 \cos^3(apk)\, ek^2 \cos(3\, fk) \cos^2(wk) \sin(wk)$$
$$+ 80 \cos^3(apk) \cos(3\, fk) \cos^2(wk) \sin(wk)$$
$$+ 120 \cos^3(apk)\, ek \cos(2\, fk) \cos^2(wk) \sin(wk)$$
$$+ 60 \cos^3(apk)\, ek^2 \cos(fk) \cos^2(wk) \sin(wk)$$
$$- 6 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^3(jk) \sin(wk)$$
$$+ 6 \sin(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^3(jk) \sin(wk)$$
$$- 30 \cos^2(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^3(jk) \sin(wk)$$
$$+ 30 \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^3(jk) \sin(wk)$$
$$- 56 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^3(jk) \sin(wk)$$
$$+ 56 \sin(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^3(jk) \sin(wk)$$
$$- 40 \cos^2(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos^3(jk) \sin(wk)$$
$$+ 40 \sin(apk) \sin(fk) \cos(2\, fk) \cos^3(jk) \sin(wk)$$
$$- 270 \cos^2(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos^3(jk) \sin(wk)$$
$$+ 270 \sin(apk)\, ek \cos(fk) \sin(fk) \cos^3(jk) \sin(wk)$$
$$- 178 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos^3(jk) \sin(wk)$$
$$+ 178 \sin(apk)\, ek^2 \sin(fk) \cos^3(jk) \sin(wk)$$
$$- 200 \cos^2(apk) \sin(apk) \sin(fk) \cos^3(jk) \sin(wk)$$
$$+ 200 \sin(apk) \sin(fk) \cos^3(jk) \sin(wk)$$
$$- 9 \cos^3(apk)\, ek^2 \cos(5\, fk) \cos^2(jk) \sin(wk)$$
$$+ 9 \cos(apk)\, ek^2 \cos(5\, fk) \cos^2(jk) \sin(wk)$$
$$- 45 \cos^3(apk)\, ek \cos(4\, fk) \cos^2(jk) \sin(wk)$$
$$+ 45 \cos(apk)\, ek \cos(4\, fk) \cos^2(jk) \sin(wk)$$

$$- 45 \cos^{3}(apk)\, ek^{2} \cos(3\, fk) \cos^{2}(jk) \sin(wk)$$

$$+ 45 \cos(apk)\, ek^{2} \cos(3\, fk) \cos^{2}(jk) \sin(wk)$$

$$- 60 \cos^{3}(apk) \cos(3\, fk) \cos^{2}(jk) \sin(wk)$$

$$+ 60 \cos(apk) \cos(3\, fk) \cos^{2}(jk) \sin(wk)$$

$$- 180 \cos^{3}(apk)\, ek \cos(2\, fk) \cos^{2}(jk) \sin(wk)$$

$$+ 180 \cos(apk)\, ek \cos(2\, fk) \cos^{2}(jk) \sin(wk)$$

$$- 90 \cos^{3}(apk)\, ek^{2} \cos(fk) \cos^{2}(jk) \sin(wk)$$

$$+ 90 \cos(apk)\, ek^{2} \cos(fk) \cos^{2}(jk) \sin(wk)$$

$$- 180 \cos^{3}(apk) \cos(fk) \cos^{2}(jk) \sin(wk)$$

$$+ 180 \cos(apk) \cos(fk) \cos^{2}(jk) \sin(wk)$$

$$- 18 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos(jk) \sin(wk)$$

$$- 90 \cos^{2}(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \sin(wk)$$

$$- 48 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos(jk) \sin(wk)$$

$$- 120 \cos^{2}(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos(jk) \sin(wk)$$

$$- 90 \cos^{2}(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \sin(wk)$$

$$+ 66 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(jk) \sin(wk)$$

$$+ 120 \cos^{2}(apk) \sin(apk) \sin(fk) \cos(jk) \sin(wk)$$

$$- 3 \cos^{3}(apk)\, ek^{2} \cos(5\, fk) \sin(wk) - 15 \cos^{3}(apk)\, ek \cos(4\, fk) \sin(wk)$$

$$+ 5 \cos^{3}(apk)\, ek^{2} \cos(3\, fk) \sin(wk) - 20 \cos^{3}(apk) \cos(3\, fk) \sin(wk)$$

$$+ 60 \cos^{3}(apk)\, ek \cos(2\, fk) \sin(wk) + 30 \cos^{3}(apk)\, ek^{2} \cos(fk) \sin(wk)$$

$$+ 180 \cos^{3}(apk) \cos(fk) \sin(wk) - 12 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(5\, fk) \cos^{3}(jk)$$

$$\cos^{3}(wk) + 12 \sin(apk)\, ek^{2} \cos(5\, fk) \cos^{3}(jk) \cos^{3}(wk)$$

$$- 60 \cos^{2}(apk) \sin(apk)\, ek \cos(4\, fk) \cos^{3}(jk) \cos^{3}(wk)$$

$$+ 60 \sin(apk)\, ek \cos(4\, fk) \cos^{3}(jk) \cos^{3}(wk)$$

$$- 40 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(3\, fk) \cos^{3}(jk) \cos^{3}(wk)$$

$$+ 40 \sin(apk)\, ek^{2} \cos(3\, fk) \cos^{3}(jk) \cos^{3}(wk)$$

$$- 80 \cos^{2}(apk) \sin(apk) \cos(3\, fk) \cos^{3}(jk) \cos^{3}(wk)$$

$$+ 80 \sin(apk) \cos(3\, fk) \cos^{3}(jk) \cos^{3}(wk)$$

$$- 120 \cos^{2}(apk) \sin(apk)\, ek \cos(2\, fk) \cos^{3}(jk) \cos^{3}(wk)$$

$$
\begin{aligned}
&+ 120 \sin(apk)\, ek \cos(2\, fk) \cos^{3}(jk) \cos^{3}(wk) \\
&- 60 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(fk) \cos^{3}(jk) \cos^{3}(wk) \\
&+ 60 \sin(apk)\, ek^{2} \cos(fk) \cos^{3}(jk) \cos^{3}(wk) \\
&+ 72 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&- 72 \cos(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&+ 360 \cos^{3}(apk)\, ek \sin(fk) \cos(3\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&- 360 \cos(apk)\, ek \sin(fk) \cos(3\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&+ 312 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&- 312 \cos(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&+ 480 \cos^{3}(apk) \sin(fk) \cos(2\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&- 480 \cos(apk) \sin(fk) \cos(2\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&+ 1080 \cos^{3}(apk)\, ek \cos(fk) \sin(fk) \cos^{2}(jk) \cos^{3}(wk) \\
&- 1080 \cos(apk)\, ek \cos(fk) \sin(fk) \cos^{2}(jk) \cos^{3}(wk) \\
&+ 336 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos^{2}(jk) \cos^{3}(wk) \\
&- 336 \cos(apk)\, ek^{2} \sin(fk) \cos^{2}(jk) \cos^{3}(wk) \\
&+ 240 \cos^{3}(apk) \sin(fk) \cos^{2}(jk) \cos^{3}(wk) \\
&- 240 \cos(apk) \sin(fk) \cos^{2}(jk) \cos^{3}(wk) \\
&- 36 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(5\, fk) \cos(jk) \cos^{3}(wk) \\
&- 180 \cos^{2}(apk) \sin(apk)\, ek \cos(4\, fk) \cos(jk) \cos^{3}(wk) \\
&- 120 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(3\, fk) \cos(jk) \cos^{3}(wk) \\
&- 240 \cos^{2}(apk) \sin(apk) \cos(3\, fk) \cos(jk) \cos^{3}(wk) \\
&- 360 \cos^{2}(apk) \sin(apk)\, ek \cos(2\, fk) \cos(jk) \cos^{3}(wk) \\
&- 180 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(fk) \cos(jk) \cos^{3}(wk) \\
&+ 24 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos^{3}(wk) \\
&+ 120 \cos^{3}(apk)\, ek \sin(fk) \cos(3\, fk) \cos^{3}(wk) \\
&+ 104 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos^{3}(wk) \\
&+ 160 \cos^{3}(apk) \sin(fk) \cos(2\, fk) \cos^{3}(wk) \\
&+ 360 \cos^{3}(apk)\, ek \cos(fk) \sin(fk) \cos^{3}(wk) \\
&+ 112 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos^{3}(wk) + 80 \cos^{3}(apk) \sin(fk) \cos^{3}(wk)
\end{aligned}
$$

$$+ 9 \cos^2(apk) \sin(apk)\, ek^2 \cos(5\, fk) \cos^3(jk) \cos(wk)$$
$$- 9 \sin(apk)\, ek^2 \cos(5\, fk) \cos^3(jk) \cos(wk)$$
$$+ 45 \cos^2(apk) \sin(apk)\, ek \cos(4\, fk) \cos^3(jk) \cos(wk)$$
$$- 45 \sin(apk)\, ek \cos(4\, fk) \cos^3(jk) \cos(wk)$$
$$+ 45 \cos^2(apk) \sin(apk)\, ek^2 \cos(3\, fk) \cos^3(jk) \cos(wk)$$
$$- 45 \sin(apk)\, ek^2 \cos(3\, fk) \cos^3(jk) \cos(wk)$$
$$+ 60 \cos^2(apk) \sin(apk) \cos(3\, fk) \cos^3(jk) \cos(wk)$$
$$- 60 \sin(apk) \cos(3\, fk) \cos^3(jk) \cos(wk)$$
$$+ 180 \cos^2(apk) \sin(apk)\, ek \cos(2\, fk) \cos^3(jk) \cos(wk)$$
$$- 180 \sin(apk)\, ek \cos(2\, fk) \cos^3(jk) \cos(wk)$$
$$+ 90 \cos^2(apk) \sin(apk)\, ek^2 \cos(fk) \cos^3(jk) \cos(wk)$$
$$- 90 \sin(apk)\, ek^2 \cos(fk) \cos^3(jk) \cos(wk)$$
$$+ 180 \cos^2(apk) \sin(apk) \cos(fk) \cos^3(jk) \cos(wk)$$
$$- 180 \sin(apk) \cos(fk) \cos^3(jk) \cos(wk)$$
$$- 54 \cos^3(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(jk) \cos(wk)$$
$$+ 54 \cos(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(jk) \cos(wk)$$
$$- 270 \cos^3(apk)\, ek \sin(fk) \cos(3\, fk) \cos^2(jk) \cos(wk)$$
$$+ 270 \cos(apk)\, ek \sin(fk) \cos(3\, fk) \cos^2(jk) \cos(wk)$$
$$- 264 \cos^3(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(jk) \cos(wk)$$
$$+ 264 \cos(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(jk) \cos(wk)$$
$$- 360 \cos^3(apk) \sin(fk) \cos(2\, fk) \cos^2(jk) \cos(wk)$$
$$+ 360 \cos(apk) \sin(fk) \cos(2\, fk) \cos^2(jk) \cos(wk)$$
$$- 990 \cos^3(apk)\, ek \cos(fk) \sin(fk) \cos^2(jk) \cos(wk)$$
$$+ 990 \cos(apk)\, ek \cos(fk) \sin(fk) \cos^2(jk) \cos(wk)$$
$$- 402 \cos^3(apk)\, ek^2 \sin(fk) \cos^2(jk) \cos(wk)$$
$$+ 402 \cos(apk)\, ek^2 \sin(fk) \cos^2(jk) \cos(wk)$$
$$- 360 \cos^3(apk) \sin(fk) \cos^2(jk) \cos(wk)$$
$$+ 360 \cos(apk) \sin(fk) \cos^2(jk) \cos(wk)$$
$$+ 27 \cos^2(apk) \sin(apk)\, ek^2 \cos(5\, fk) \cos(jk) \cos(wk)$$

```
         2
+ 135 cos (apk) sin(apk) ek cos(4 fk) cos(jk) cos(wk)
        2                 2
+ 75 cos (apk) sin(apk) ek  cos(3 fk) cos(jk) cos(wk)
         2
+ 180 cos (apk) sin(apk) cos(3 fk) cos(jk) cos(wk)
         2
+ 180 cos (apk) sin(apk) ek cos(2 fk) cos(jk) cos(wk)
        2                 2
+ 90 cos (apk) sin(apk) ek  cos(fk) cos(jk) cos(wk)
         2
- 180 cos (apk) sin(apk) cos(fk) cos(jk) cos(wk)
        3        2
- 18 cos (apk) ek  sin(fk) cos(4 fk) cos(wk)
        3
- 90 cos (apk) ek sin(fk) cos(3 fk) cos(wk)
        3        2
- 48 cos (apk) ek  sin(fk) cos(2 fk) cos(wk)
         3
- 120 cos (apk) sin(fk) cos(2 fk) cos(wk)
        3                                         3         2
- 90 cos (apk) ek cos(fk) sin(fk) cos(wk) + 66 cos (apk) ek  sin(fk) cos(wk)
         3
+ 120 cos (apk) sin(fk) cos(wk)) zb2/(240 nk)
```

$\delta(za2S2^3) =$

```
              3        2                  3        2
     n (24 cos (apk) ek  sin(fk) cos(4 fk) cos (jk) cos (wk) sin(wk)
         3                              3        2
+ 120 cos (apk) ek sin(fk) cos(3 fk) cos (jk) cos (wk) sin(wk)
         3        2                       3        2
+ 104 cos (apk) ek  sin(fk) cos(2 fk) cos (jk) cos (wk) sin(wk)
         3                           3        2
+ 160 cos (apk) sin(fk) cos(2 fk) cos (jk) cos (wk) sin(wk)
         3                            3        2
+ 360 cos (apk) ek cos(fk) sin(fk) cos (jk) cos (wk) sin(wk)
         3        2            3        2
+ 112 cos (apk) ek  sin(fk) cos (jk) cos (wk) sin(wk)
        3                 3        2
+ 80 cos (apk) sin(fk) cos (jk) cos (wk) sin(wk)
        2                 2              2        2
- 36 cos (apk) sin(apk) ek  cos(5 fk) cos (jk) cos (wk) sin(wk)
         2                              2        2
- 180 cos (apk) sin(apk) ek cos(4 fk) cos (jk) cos (wk) sin(wk)
         2                 2              2        2
- 120 cos (apk) sin(apk) ek  cos(3 fk) cos (jk) cos (wk) sin(wk)
         2                           2        2
- 240 cos (apk) sin(apk) cos(3 fk) cos (jk) cos (wk) sin(wk)
         2                              2        2
- 360 cos (apk) sin(apk) ek cos(2 fk) cos (jk) cos (wk) sin(wk)
         2                 2            2        2
- 180 cos (apk) sin(apk) ek  cos(fk) cos (jk) cos (wk) sin(wk)
        3        2                               2
+ 72 cos (apk) ek  sin(fk) cos(4 fk) cos(jk) cos (wk) sin(wk)
```

$$
\begin{aligned}
&- 72 \cos(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)\\
&+ 360 \cos^{3}(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)\\
&- 360 \cos(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)\\
&+ 312 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)\\
&- 312 \cos(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)\\
&+ 480 \cos^{3}(apk) \sin(fk) \cos(2\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)\\
&- 480 \cos(apk) \sin(fk) \cos(2\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)\\
&+ 1080 \cos^{3}(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \cos^{2}(wk) \sin(wk)\\
&- 1080 \cos(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \cos^{2}(wk) \sin(wk)\\
&+ 336 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos(jk) \cos^{2}(wk) \sin(wk)\\
&- 336 \cos(apk)\, ek^{2} \sin(fk) \cos(jk) \cos^{2}(wk) \sin(wk)\\
&+ 240 \cos^{3}(apk) \sin(fk) \cos(jk) \cos^{2}(wk) \sin(wk)\\
&- 240 \cos(apk) \sin(fk) \cos(jk) \cos^{2}(wk) \sin(wk)\\
&- 12 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(5\, fk) \cos^{2}(wk) \sin(wk)\\
&+ 12 \sin(apk)\, ek^{2} \cos(5\, fk) \cos^{2}(wk) \sin(wk)\\
&- 60 \cos^{2}(apk) \sin(apk)\, ek \cos(4\, fk) \cos^{2}(wk) \sin(wk)\\
&+ 60 \sin(apk)\, ek \cos(4\, fk) \cos^{2}(wk) \sin(wk)\\
&- 40 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(3\, fk) \cos^{2}(wk) \sin(wk)\\
&+ 40 \sin(apk)\, ek^{2} \cos(3\, fk) \cos^{2}(wk) \sin(wk)\\
&- 80 \cos^{2}(apk) \sin(apk) \cos(3\, fk) \cos^{2}(wk) \sin(wk)\\
&+ 80 \sin(apk) \cos(3\, fk) \cos^{2}(wk) \sin(wk)\\
&- 120 \cos^{2}(apk) \sin(apk)\, ek \cos(2\, fk) \cos^{2}(wk) \sin(wk)\\
&+ 120 \sin(apk)\, ek \cos(2\, fk) \cos^{2}(wk) \sin(wk)\\
&- 60 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(fk) \cos^{2}(wk) \sin(wk)\\
&+ 60 \sin(apk)\, ek^{2} \cos(fk) \cos^{2}(wk) \sin(wk)\\
&- 6 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos^{3}(jk) \sin(wk)\\
&- 30 \cos^{3}(apk)\, ek \sin(fk) \cos(3\, fk) \cos^{3}(jk) \sin(wk)\\
&- 56 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos^{3}(jk) \sin(wk)\\
&- 40 \cos^{3}(apk) \sin(fk) \cos(2\, fk) \cos^{3}(jk) \sin(wk)
\end{aligned}
$$

$$- 270 \cos^3(apk)\, ek \cos(fk) \sin(fk) \cos^3(jk) \sin(wk)$$
$$- 178 \cos^3(apk)\, ek^2 \sin(fk) \cos^3(jk) \sin(wk)$$
$$- 200 \cos^3(apk) \sin(fk) \cos^3(jk) \sin(wk)$$
$$+ 9 \cos^2(apk) \sin(apk)\, ek^2 \cos(5\, fk) \cos^2(jk) \sin(wk)$$
$$+ 45 \cos^2(apk) \sin(apk)\, ek \cos(4\, fk) \cos^2(jk) \sin(wk)$$
$$+ 45 \cos^2(apk) \sin(apk)\, ek^2 \cos(3\, fk) \cos^2(jk) \sin(wk)$$
$$+ 60 \cos^2(apk) \sin(apk) \cos(3\, fk) \cos^2(jk) \sin(wk)$$
$$+ 180 \cos^2(apk) \sin(apk)\, ek \cos(2\, fk) \cos^2(jk) \sin(wk)$$
$$+ 90 \cos^2(apk) \sin(apk)\, ek^2 \cos(fk) \cos^2(jk) \sin(wk)$$
$$+ 180 \cos^2(apk) \sin(apk) \cos(fk) \cos^2(jk) \sin(wk)$$
$$- 18 \cos^3(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos(jk) \sin(wk)$$
$$+ 18 \cos(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos(jk) \sin(wk)$$
$$- 90 \cos^3(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \sin(wk)$$
$$+ 90 \cos(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \sin(wk)$$
$$- 48 \cos^3(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos(jk) \sin(wk)$$
$$+ 48 \cos(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos(jk) \sin(wk)$$
$$- 120 \cos^3(apk) \sin(fk) \cos(2\, fk) \cos(jk) \sin(wk)$$
$$+ 120 \cos(apk) \sin(fk) \cos(2\, fk) \cos(jk) \sin(wk)$$
$$- 90 \cos^3(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \sin(wk)$$
$$+ 90 \cos(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \sin(wk)$$
$$+ 66 \cos^3(apk)\, ek^2 \sin(fk) \cos(jk) \sin(wk)$$
$$- 66 \cos(apk)\, ek^2 \sin(fk) \cos(jk) \sin(wk)$$
$$+ 120 \cos^3(apk) \sin(fk) \cos(jk) \sin(wk) - 120 \cos(apk) \sin(fk) \cos(jk) \sin(wk)$$
$$+ 3 \cos^2(apk) \sin(apk)\, ek^2 \cos(5\, fk) \sin(wk)$$
$$- 3 \sin(apk)\, ek^2 \cos(5\, fk) \sin(wk) + 15 \cos^2(apk) \sin(apk)\, ek \cos(4\, fk)$$
$$\sin(wk) - 15 \sin(apk)\, ek \cos(4\, fk) \sin(wk)$$
$$- 5 \cos^2(apk) \sin(apk)\, ek^2 \cos(3\, fk) \sin(wk)$$
$$+ 5 \sin(apk)\, ek^2 \cos(3\, fk) \sin(wk) + 20 \cos^2(apk) \sin(apk) \cos(3\, fk) \sin(wk)$$
$$- 20 \sin(apk) \cos(3\, fk) \sin(wk) - 60 \cos^2(apk) \sin(apk)\, ek \cos(2\, fk) \sin(wk)$$

$$+ 60 \sin(apk)\, ek \cos(2\, fk) \sin(wk) - 30 \cos^2(apk) \sin(apk)\, ek^2 \cos(fk) \sin(wk)$$
$$+ 30 \sin(apk)\, ek^2 \cos(fk) \sin(wk) - 180 \cos^2(apk) \sin(apk) \cos(fk) \sin(wk)$$
$$+ 180 \sin(apk) \cos(fk) \sin(wk) - 12 \cos^3(apk)\, ek^2 \cos(5\, fk) \cos^3(jk) \cos^3(wk)$$
$$- 60 \cos^3(apk)\, ek \cos(4\, fk) \cos^3(jk) \cos^3(wk)$$
$$- 40 \cos^3(apk)\, ek^2 \cos(3\, fk) \cos^3(jk) \cos^3(wk)$$
$$- 80 \cos^3(apk) \cos(3\, fk) \cos^3(jk) \cos^3(wk)$$
$$- 120 \cos^3(apk)\, ek \cos(2\, fk) \cos^3(jk) \cos^3(wk)$$
$$- 60 \cos^3(apk)\, ek^2 \cos(fk) \cos^3(jk) \cos^3(wk)$$
$$- 72 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(jk) \cos^3(wk)$$
$$- 360 \cos^2(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^2(jk) \cos^3(wk)$$
$$- 312 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(jk) \cos^3(wk)$$
$$- 480 \cos^2(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos^2(jk) \cos^3(wk)$$
$$- 1080 \cos^2(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos^2(jk) \cos^3(wk)$$
$$- 336 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos^2(jk) \cos^3(wk)$$
$$- 240 \cos^2(apk) \sin(apk) \sin(fk) \cos^2(jk) \cos^3(wk)$$
$$- 36 \cos^3(apk)\, ek^2 \cos(5\, fk) \cos(jk) \cos^3(wk)$$
$$+ 36 \cos(apk)\, ek^2 \cos(5\, fk) \cos(jk) \cos^3(wk)$$
$$- 180 \cos^3(apk)\, ek \cos(4\, fk) \cos(jk) \cos^3(wk)$$
$$+ 180 \cos(apk)\, ek \cos(4\, fk) \cos(jk) \cos^3(wk)$$
$$- 120 \cos^3(apk)\, ek^2 \cos(3\, fk) \cos(jk) \cos^3(wk)$$
$$+ 120 \cos(apk)\, ek^2 \cos(3\, fk) \cos(jk) \cos^3(wk)$$
$$- 240 \cos^3(apk) \cos(3\, fk) \cos(jk) \cos^3(wk)$$
$$+ 240 \cos(apk) \cos(3\, fk) \cos(jk) \cos^3(wk)$$
$$- 360 \cos^3(apk)\, ek \cos(2\, fk) \cos(jk) \cos^3(wk)$$
$$+ 360 \cos(apk)\, ek \cos(2\, fk) \cos(jk) \cos^3(wk)$$
$$- 180 \cos^3(apk)\, ek^2 \cos(fk) \cos(jk) \cos^3(wk)$$
$$+ 180 \cos(apk)\, ek^2 \cos(fk) \cos(jk) \cos^3(wk)$$
$$- 24 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^3(wk)$$
$$+ 24 \sin(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^3(wk)$$

$$- 120 \cos^2(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^3(wk)$$
$$+ 120 \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^3(wk)$$
$$- 104 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^3(wk)$$
$$+ 104 \sin(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^3(wk)$$
$$- 160 \cos^2(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos^3(wk)$$
$$+ 160 \sin(apk) \sin(fk) \cos(2\, fk) \cos^3(wk)$$
$$- 360 \cos^2(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos^3(wk)$$
$$+ 360 \sin(apk)\, ek \cos(fk) \sin(fk) \cos^3(wk)$$
$$- 112 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos^3(wk)$$
$$+ 112 \sin(apk)\, ek^2 \sin(fk) \cos^3(wk) - 80 \cos^2(apk) \sin(apk) \sin(fk) \cos^3(wk)$$
$$+ 80 \sin(apk) \sin(fk) \cos^3(wk) + 9 \cos^3(apk)\, ek^2 \cos(5\, fk) \cos^3(jk) \cos(wk)$$
$$+ 45 \cos^3(apk)\, ek \cos(4\, fk) \cos^3(jk) \cos(wk)$$
$$+ 45 \cos^3(apk)\, ek^2 \cos(3\, fk) \cos^3(jk) \cos(wk)$$
$$+ 60 \cos^3(apk) \cos(3\, fk) \cos^3(jk) \cos(wk)$$
$$+ 180 \cos^3(apk)\, ek \cos(2\, fk) \cos^3(jk) \cos(wk)$$
$$+ 90 \cos^3(apk)\, ek^2 \cos(fk) \cos^3(jk) \cos(wk)$$
$$+ 180 \cos^3(apk) \cos(fk) \cos^3(jk) \cos(wk)$$
$$+ 54 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(jk) \cos(wk)$$
$$+ 270 \cos^2(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^2(jk) \cos(wk)$$
$$+ 264 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(jk) \cos(wk)$$
$$+ 360 \cos^2(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos^2(jk) \cos(wk)$$
$$+ 990 \cos^2(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos^2(jk) \cos(wk)$$
$$+ 402 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos^2(jk) \cos(wk)$$
$$+ 360 \cos^2(apk) \sin(apk) \sin(fk) \cos^2(jk) \cos(wk)$$
$$+ 27 \cos^3(apk)\, ek^2 \cos(5\, fk) \cos(jk) \cos(wk)$$
$$- 27 \cos(apk)\, ek^2 \cos(5\, fk) \cos(jk) \cos(wk)$$
$$+ 135 \cos^3(apk)\, ek \cos(4\, fk) \cos(jk) \cos(wk)$$
$$- 135 \cos(apk)\, ek \cos(4\, fk) \cos(jk) \cos(wk)$$
$$+ 75 \cos^3(apk)\, ek^2 \cos(3\, fk) \cos(jk) \cos(wk)$$

$$
\begin{aligned}
&- 75 \cos(apk)\, ek^{2} \cos(3\, fk) \cos(jk) \cos(wk) \\
&+ 180 \cos^{3}(apk) \cos(3\, fk) \cos(jk) \cos(wk) \\
&- 180 \cos(apk) \cos(3\, fk) \cos(jk) \cos(wk) \\
&+ 180 \cos^{3}(apk)\, ek \cos(2\, fk) \cos(jk) \cos(wk) \\
&- 180 \cos(apk)\, ek \cos(2\, fk) \cos(jk) \cos(wk) \\
&+ 90 \cos^{3}(apk)\, ek^{2} \cos(fk) \cos(jk) \cos(wk) \\
&- 90 \cos(apk)\, ek^{2} \cos(fk) \cos(jk) \cos(wk) \\
&- 180 \cos^{3}(apk) \cos(fk) \cos(jk) \cos(wk) + 180 \cos(apk) \cos(fk) \cos(jk) \cos(wk) \\
&+ 18 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos(wk) \\
&- 18 \sin(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos(wk) \\
&+ 90 \cos^{2}(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos(wk) \\
&- 90 \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos(wk) \\
&+ 48 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos(wk) \\
&- 48 \sin(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos(wk) \\
&+ 120 \cos^{2}(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos(wk) \\
&- 120 \sin(apk) \sin(fk) \cos(2\, fk) \cos(wk) \\
&+ 90 \cos^{2}(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos(wk) \\
&- 90 \sin(apk)\, ek \cos(fk) \sin(fk) \cos(wk) \\
&- 66 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(wk) + 66 \sin(apk)\, ek^{2} \sin(fk) \cos(wk) \\
&- 120 \cos^{2}(apk) \sin(apk) \sin(fk) \cos(wk) + 120 \sin(apk) \sin(fk) \cos(wk))\, zb2 \\
&/(240\, nk)
\end{aligned}
$$

$$\delta(za2S1S2^{2}) =$$

$$
\begin{aligned}
&n\, (24 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos^{3}(jk) \cos^{2}(wk) \sin(wk) \\
&+ 120 \cos^{2}(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^{3}(jk) \cos^{2}(wk) \sin(wk) \\
&+ 104 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos^{3}(jk) \cos^{2}(wk) \sin(wk) \\
&+ 160 \cos^{2}(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos^{3}(jk) \cos^{2}(wk) \sin(wk) \\
&+ 360 \cos^{2}(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos^{3}(jk) \cos^{2}(wk) \sin(wk)
\end{aligned}
$$

$$+ 112 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos^{3}(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 80 \cos^{2}(apk) \sin(apk) \sin(fk) \cos^{3}(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 36 \cos^{3}(apk)\, ek^{2} \cos(5\, fk) \cos^{2}(jk) \cos^{2}(wk) \sin(wk)$$

$$- 24 \cos(apk)\, ek^{2} \cos(5\, fk) \cos^{2}(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 180 \cos^{3}(apk)\, ek \cos(4\, fk) \cos^{2}(jk) \cos^{2}(wk) \sin(wk)$$

$$- 120 \cos(apk)\, ek \cos(4\, fk) \cos^{2}(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 120 \cos^{3}(apk)\, ek^{2} \cos(3\, fk) \cos^{2}(jk) \cos^{2}(wk) \sin(wk)$$

$$- 80 \cos(apk)\, ek^{2} \cos(3\, fk) \cos^{2}(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 240 \cos^{3}(apk) \cos(3\, fk) \cos^{2}(jk) \cos^{2}(wk) \sin(wk)$$

$$- 160 \cos(apk) \cos(3\, fk) \cos^{2}(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 360 \cos^{3}(apk)\, ek \cos(2\, fk) \cos^{2}(jk) \cos^{2}(wk) \sin(wk)$$

$$- 240 \cos(apk)\, ek \cos(2\, fk) \cos^{2}(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 180 \cos^{3}(apk)\, ek^{2} \cos(fk) \cos^{2}(jk) \cos^{2}(wk) \sin(wk)$$

$$- 120 \cos(apk)\, ek^{2} \cos(fk) \cos^{2}(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 72 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$

$$- 24 \sin(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 360 \cos^{2}(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$

$$- 120 \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 312 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$

$$- 104 \sin(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 480 \cos^{2}(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$

$$- 160 \sin(apk) \sin(fk) \cos(2\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 1080 \cos^{2}(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$

$$- 360 \sin(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 336 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$

$$- 112 \sin(apk)\, ek^{2} \sin(fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 240 \cos^{2}(apk) \sin(apk) \sin(fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$

$$- 80 \sin(apk) \sin(fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$

$$+ 12 \cos^{3}(apk)\, ek^{2} \cos(5\, fk) \cos^{2}(wk) \sin(wk)$$

$$
\begin{aligned}
& - 12 \cos(apk)\, ek^{2} \cos(5\, fk) \cos^{2}(wk) \sin(wk) \\
& + 60 \cos^{3}(apk)\, ek \cos(4\, fk) \cos^{2}(wk) \sin(wk) \\
& - 60 \cos(apk)\, ek \cos(4\, fk) \cos^{2}(wk) \sin(wk) \\
& + 40 \cos^{3}(apk)\, ek^{2} \cos(3\, fk) \cos^{2}(wk) \sin(wk) \\
& - 40 \cos(apk)\, ek^{2} \cos(3\, fk) \cos^{2}(wk) \sin(wk) \\
& + 80 \cos^{3}(apk) \cos(3\, fk) \cos^{2}(wk) \sin(wk) \\
& - 80 \cos(apk) \cos(3\, fk) \cos^{2}(wk) \sin(wk) \\
& + 120 \cos^{3}(apk)\, ek \cos(2\, fk) \cos^{2}(wk) \sin(wk) \\
& - 120 \cos(apk)\, ek \cos(2\, fk) \cos^{2}(wk) \sin(wk) \\
& + 60 \cos^{3}(apk)\, ek^{2} \cos(fk) \cos^{2}(wk) \sin(wk) \\
& - 60 \cos(apk)\, ek^{2} \cos(fk) \cos^{2}(wk) \sin(wk) \\
& - 6 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos^{3}(jk) \sin(wk) \\
& - 30 \cos^{2}(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^{3}(jk) \sin(wk) \\
& - 56 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos^{3}(jk) \sin(wk) \\
& - 40 \cos^{2}(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos^{3}(jk) \sin(wk) \\
& - 270 \cos^{2}(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos^{3}(jk) \sin(wk) \\
& - 178 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos^{3}(jk) \sin(wk) \\
& - 200 \cos^{2}(apk) \sin(apk) \sin(fk) \cos^{3}(jk) \sin(wk) \\
& - 9 \cos^{3}(apk)\, ek^{2} \cos(5\, fk) \cos^{2}(jk) \sin(wk) \\
& + 6 \cos(apk)\, ek^{2} \cos(5\, fk) \cos^{2}(jk) \sin(wk) \\
& - 45 \cos^{3}(apk)\, ek \cos(4\, fk) \cos^{2}(jk) \sin(wk) \\
& + 30 \cos(apk)\, ek \cos(4\, fk) \cos^{2}(jk) \sin(wk) \\
& - 45 \cos^{3}(apk)\, ek^{2} \cos(3\, fk) \cos^{2}(jk) \sin(wk) \\
& + 30 \cos(apk)\, ek^{2} \cos(3\, fk) \cos^{2}(jk) \sin(wk) \\
& - 60 \cos^{3}(apk) \cos(3\, fk) \cos^{2}(jk) \sin(wk) \\
& + 40 \cos(apk) \cos(3\, fk) \cos^{2}(jk) \sin(wk) \\
& - 180 \cos^{3}(apk)\, ek \cos(2\, fk) \cos^{2}(jk) \sin(wk) \\
& + 120 \cos(apk)\, ek \cos(2\, fk) \cos^{2}(jk) \sin(wk) \\
& - 90 \cos^{3}(apk)\, ek^{2} \cos(fk) \cos^{2}(jk) \sin(wk)
\end{aligned}
$$

```
                2            2
+ 60 cos(apk) ek  cos(fk) cos (jk) sin(wk)
         3                 2
- 180 cos (apk) cos(fk) cos (jk) sin(wk)
                          2
+ 120 cos(apk) cos(fk) cos (jk) sin(wk)
        2                 2
- 18 cos (apk) sin(apk) ek  sin(fk) cos(4 fk) cos(jk) sin(wk)
               2
+ 6 sin(apk) ek  sin(fk) cos(4 fk) cos(jk) sin(wk)
        2
- 90 cos (apk) sin(apk) ek sin(fk) cos(3 fk) cos(jk) sin(wk)

+ 30 sin(apk) ek sin(fk) cos(3 fk) cos(jk) sin(wk)
        2                 2
- 48 cos (apk) sin(apk) ek  sin(fk) cos(2 fk) cos(jk) sin(wk)
                2
+ 16 sin(apk) ek  sin(fk) cos(2 fk) cos(jk) sin(wk)
         2
- 120 cos (apk) sin(apk) sin(fk) cos(2 fk) cos(jk) sin(wk)

+ 40 sin(apk) sin(fk) cos(2 fk) cos(jk) sin(wk)
        2
- 90 cos (apk) sin(apk) ek cos(fk) sin(fk) cos(jk) sin(wk)

+ 30 sin(apk) ek cos(fk) sin(fk) cos(jk) sin(wk)
        2                 2
+ 66 cos (apk) sin(apk) ek  sin(fk) cos(jk) sin(wk)
                2
- 22 sin(apk) ek  sin(fk) cos(jk) sin(wk)
         2
+ 120 cos (apk) sin(apk) sin(fk) cos(jk) sin(wk)
                                             3        2
- 40 sin(apk) sin(fk) cos(jk) sin(wk) - 3 cos (apk) ek  cos(5 fk) sin(wk)
               2                           3
+ 3 cos(apk) ek  cos(5 fk) sin(wk) - 15 cos (apk) ek cos(4 fk) sin(wk)
                                          3        2
+ 15 cos(apk) ek cos(4 fk) sin(wk) + 5 cos (apk) ek  cos(3 fk) sin(wk)
               2                           3
- 5 cos(apk) ek  cos(3 fk) sin(wk) - 20 cos (apk) cos(3 fk) sin(wk)
                                        3
+ 20 cos(apk) cos(3 fk) sin(wk) + 60 cos (apk) ek cos(2 fk) sin(wk)
                                           3        2
- 60 cos(apk) ek cos(2 fk) sin(wk) + 30 cos (apk) ek  cos(fk) sin(wk)
                2                          3
- 30 cos(apk) ek  cos(fk) sin(wk) + 180 cos (apk) cos(fk) sin(wk)
                                       2                 2              3
- 180 cos(apk) cos(fk) sin(wk) - 12 cos (apk) sin(apk) ek  cos(5 fk) cos (jk)
   3             2                               3        3
cos (wk) - 60 cos (apk) sin(apk) ek cos(4 fk) cos (jk) cos (wk)
        2                 2              3        3
- 40 cos (apk) sin(apk) ek  cos(3 fk) cos (jk) cos (wk)
        2                            3        3
- 80 cos (apk) sin(apk) cos(3 fk) cos (jk) cos (wk)
         2                               3        3
- 120 cos (apk) sin(apk) ek cos(2 fk) cos (jk) cos (wk)
        2                 2            3        3
- 60 cos (apk) sin(apk) ek  cos(fk) cos (jk) cos (wk)
```

$$
\begin{aligned}
&+ 72 \cos^3(apk)\, ek^2 \sin(fk) \cos(4\,fk) \cos^2(jk) \cos^3(wk) \\
&- 48 \cos(apk)\, ek^2 \sin(fk) \cos(4\,fk) \cos^2(jk) \cos^3(wk) \\
&+ 360 \cos^3(apk)\, ek \sin(fk) \cos(3\,fk) \cos^2(jk) \cos^3(wk) \\
&- 240 \cos(apk)\, ek \sin(fk) \cos(3\,fk) \cos^2(jk) \cos^3(wk) \\
&+ 312 \cos^3(apk)\, ek^2 \sin(fk) \cos(2\,fk) \cos^2(jk) \cos^3(wk) \\
&- 208 \cos(apk)\, ek^2 \sin(fk) \cos(2\,fk) \cos^2(jk) \cos^3(wk) \\
&+ 480 \cos^3(apk) \sin(fk) \cos(2\,fk) \cos^2(jk) \cos^3(wk) \\
&- 320 \cos(apk) \sin(fk) \cos(2\,fk) \cos^2(jk) \cos^3(wk) \\
&+ 1080 \cos^3(apk)\, ek \cos(fk) \sin(fk) \cos^2(jk) \cos^3(wk) \\
&- 720 \cos(apk)\, ek \cos(fk) \sin(fk) \cos^2(jk) \cos^3(wk) \\
&+ 336 \cos^3(apk)\, ek^2 \sin(fk) \cos^2(jk) \cos^3(wk) \\
&- 224 \cos(apk)\, ek^2 \sin(fk) \cos^2(jk) \cos^3(wk) \\
&+ 240 \cos^3(apk) \sin(fk) \cos^2(jk) \cos^3(wk) \\
&- 160 \cos(apk) \sin(fk) \cos^2(jk) \cos^3(wk) \\
&- 36 \cos^2(apk) \sin(apk)\, ek^2 \cos(5\,fk) \cos(jk) \cos^3(wk) \\
&+ 12 \sin(apk)\, ek^2 \cos(5\,fk) \cos(jk) \cos^3(wk) \\
&- 180 \cos^2(apk) \sin(apk)\, ek \cos(4\,fk) \cos(jk) \cos^3(wk) \\
&+ 60 \sin(apk)\, ek \cos(4\,fk) \cos(jk) \cos^3(wk) \\
&- 120 \cos^2(apk) \sin(apk)\, ek^2 \cos(3\,fk) \cos(jk) \cos^3(wk) \\
&+ 40 \sin(apk)\, ek^2 \cos(3\,fk) \cos(jk) \cos^3(wk) \\
&- 240 \cos^2(apk) \sin(apk) \cos(3\,fk) \cos(jk) \cos^3(wk) \\
&+ 80 \sin(apk) \cos(3\,fk) \cos(jk) \cos^3(wk) \\
&- 360 \cos^2(apk) \sin(apk)\, ek \cos(2\,fk) \cos(jk) \cos^3(wk) \\
&+ 120 \sin(apk)\, ek \cos(2\,fk) \cos(jk) \cos^3(wk) \\
&- 180 \cos^2(apk) \sin(apk)\, ek^2 \cos(fk) \cos(jk) \cos^3(wk) \\
&+ 60 \sin(apk)\, ek^2 \cos(fk) \cos(jk) \cos^3(wk) \\
&+ 24 \cos^3(apk)\, ek^2 \sin(fk) \cos(4\,fk) \cos^3(wk) \\
&- 24 \cos(apk)\, ek^2 \sin(fk) \cos(4\,fk) \cos^3(wk) \\
&+ 120 \cos^3(apk)\, ek \sin(fk) \cos(3\,fk) \cos^3(wk)
\end{aligned}
$$

$$- 120 \cos(apk)\, ek \sin(fk) \cos(3\, fk) \cos^{3}(wk)$$
$$+ 104 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos^{3}(wk)$$
$$- 104 \cos(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos^{3}(wk)$$
$$+ 160 \cos^{3}(apk) \sin(fk) \cos(2\, fk) \cos^{3}(wk)$$
$$- 160 \cos(apk) \sin(fk) \cos(2\, fk) \cos^{3}(wk)$$
$$+ 360 \cos^{3}(apk)\, ek \cos(fk) \sin(fk) \cos^{3}(wk)$$
$$- 360 \cos(apk)\, ek \cos(fk) \sin(fk) \cos^{3}(wk)$$
$$+ 112 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos^{3}(wk) - 112 \cos(apk)\, ek^{2} \sin(fk) \cos^{3}(wk)$$
$$+ 80 \cos^{3}(apk) \sin(fk) \cos^{3}(wk) - 80 \cos(apk) \sin(fk) \cos^{3}(wk)$$
$$+ 9 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(5\, fk) \cos^{3}(jk) \cos(wk)$$
$$+ 45 \cos^{2}(apk) \sin(apk)\, ek \cos(4\, fk) \cos^{3}(jk) \cos(wk)$$
$$+ 45 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(3\, fk) \cos^{3}(jk) \cos(wk)$$
$$+ 60 \cos^{2}(apk) \sin(apk) \cos(3\, fk) \cos^{3}(jk) \cos(wk)$$
$$+ 180 \cos^{2}(apk) \sin(apk)\, ek \cos(2\, fk) \cos^{3}(jk) \cos(wk)$$
$$+ 90 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(fk) \cos^{3}(jk) \cos(wk)$$
$$+ 180 \cos^{2}(apk) \sin(apk) \cos(fk) \cos^{3}(jk) \cos(wk)$$
$$- 54 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos^{2}(jk) \cos(wk)$$
$$+ 36 \cos(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos^{2}(jk) \cos(wk)$$
$$- 270 \cos^{3}(apk)\, ek \sin(fk) \cos(3\, fk) \cos^{2}(jk) \cos(wk)$$
$$+ 180 \cos(apk)\, ek \sin(fk) \cos(3\, fk) \cos^{2}(jk) \cos(wk)$$
$$- 264 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos^{2}(jk) \cos(wk)$$
$$+ 176 \cos(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos^{2}(jk) \cos(wk)$$
$$- 360 \cos^{3}(apk) \sin(fk) \cos(2\, fk) \cos^{2}(jk) \cos(wk)$$
$$+ 240 \cos(apk) \sin(fk) \cos(2\, fk) \cos^{2}(jk) \cos(wk)$$
$$- 990 \cos^{3}(apk)\, ek \cos(fk) \sin(fk) \cos^{2}(jk) \cos(wk)$$
$$+ 660 \cos(apk)\, ek \cos(fk) \sin(fk) \cos^{2}(jk) \cos(wk)$$
$$- 402 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos^{2}(jk) \cos(wk)$$
$$+ 268 \cos(apk)\, ek^{2} \sin(fk) \cos^{2}(jk) \cos(wk)$$
$$- 360 \cos^{3}(apk) \sin(fk) \cos^{2}(jk) \cos(wk)$$

$$+ 240 \cos(apk) \sin(fk) \cos^{2}(jk) \cos(wk)$$

$$+ 27 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(5\, fk) \cos(jk) \cos(wk)$$

$$- 9 \sin(apk)\, ek^{2} \cos(5\, fk) \cos(jk) \cos(wk)$$

$$+ 135 \cos^{2}(apk) \sin(apk)\, ek \cos(4\, fk) \cos(jk) \cos(wk)$$

$$- 45 \sin(apk)\, ek \cos(4\, fk) \cos(jk) \cos(wk)$$

$$+ 75 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(3\, fk) \cos(jk) \cos(wk)$$

$$- 25 \sin(apk)\, ek^{2} \cos(3\, fk) \cos(jk) \cos(wk)$$

$$+ 180 \cos^{2}(apk) \sin(apk) \cos(3\, fk) \cos(jk) \cos(wk)$$

$$- 60 \sin(apk) \cos(3\, fk) \cos(jk) \cos(wk)$$

$$+ 180 \cos^{2}(apk) \sin(apk)\, ek \cos(2\, fk) \cos(jk) \cos(wk)$$

$$- 60 \sin(apk)\, ek \cos(2\, fk) \cos(jk) \cos(wk)$$

$$+ 90 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(fk) \cos(jk) \cos(wk)$$

$$- 30 \sin(apk)\, ek^{2} \cos(fk) \cos(jk) \cos(wk)$$

$$- 180 \cos^{2}(apk) \sin(apk) \cos(fk) \cos(jk) \cos(wk)$$

$$+ 60 \sin(apk) \cos(fk) \cos(jk) \cos(wk) - 18 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos(4\, fk)$$

$$\cos(wk) + 18 \cos(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos(wk)$$

$$- 90 \cos^{3}(apk)\, ek \sin(fk) \cos(3\, fk) \cos(wk)$$

$$+ 90 \cos(apk)\, ek \sin(fk) \cos(3\, fk) \cos(wk)$$

$$- 48 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos(wk)$$

$$+ 48 \cos(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos(wk)$$

$$- 120 \cos^{3}(apk) \sin(fk) \cos(2\, fk) \cos(wk)$$

$$+ 120 \cos(apk) \sin(fk) \cos(2\, fk) \cos(wk)$$

$$- 90 \cos^{3}(apk)\, ek \cos(fk) \sin(fk) \cos(wk)$$

$$+ 90 \cos(apk)\, ek \cos(fk) \sin(fk) \cos(wk) + 66 \cos^{3}(apk)\, ek^{2} \sin(fk) \cos(wk)$$

$$- 66 \cos(apk)\, ek^{2} \sin(fk) \cos(wk) + 120 \cos^{3}(apk) \sin(fk) \cos(wk)$$

$$- 120 \cos(apk) \sin(fk) \cos(wk))\, zb2/(240\, nk)$$

$\delta(za2S2S1^2) =$

$$
\begin{aligned}
&- n\,(24 \cos^3(apk)\, ek^2 \sin(fk) \cos(4\,fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&- 24 \cos(apk)\, ek^2 \sin(fk) \cos(4\,fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&+ 120 \cos^3(apk)\, ek \sin(fk) \cos(3\,fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&- 120 \cos(apk)\, ek \sin(fk) \cos(3\,fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&+ 104 \cos^3(apk)\, ek^2 \sin(fk) \cos(2\,fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&- 104 \cos(apk)\, ek^2 \sin(fk) \cos(2\,fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&+ 160 \cos^3(apk) \sin(fk) \cos(2\,fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&- 160 \cos(apk) \sin(fk) \cos(2\,fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&+ 360 \cos^3(apk)\, ek \cos(fk) \sin(fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&- 360 \cos(apk)\, ek \cos(fk) \sin(fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&+ 112 \cos^3(apk)\, ek^2 \sin(fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&- 112 \cos(apk)\, ek^2 \sin(fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&+ 80 \cos^3(apk) \sin(fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&- 80 \cos(apk) \sin(fk) \cos^3(jk) \cos^2(wk) \sin(wk) \\
&- 36 \cos^2(apk) \sin(apk)\, ek^2 \cos(5\,fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 12 \sin(apk)\, ek^2 \cos(5\,fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&- 180 \cos^2(apk) \sin(apk)\, ek \cos(4\,fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 60 \sin(apk)\, ek \cos(4\,fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&- 120 \cos^2(apk) \sin(apk)\, ek^2 \cos(3\,fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 40 \sin(apk)\, ek^2 \cos(3\,fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&- 240 \cos^2(apk) \sin(apk) \cos(3\,fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 80 \sin(apk) \cos(3\,fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&- 360 \cos^2(apk) \sin(apk)\, ek \cos(2\,fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 120 \sin(apk)\, ek \cos(2\,fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&- 180 \cos^2(apk) \sin(apk)\, ek^2 \cos(fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 60 \sin(apk)\, ek^2 \cos(fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 72 \cos^3(apk)\, ek^2 \sin(fk) \cos(4\,fk) \cos(jk) \cos^2(wk) \sin(wk) \\
&- 48 \cos(apk)\, ek^2 \sin(fk) \cos(4\,fk) \cos(jk) \cos^2(wk) \sin(wk)
\end{aligned}
$$

$$+ 360 \cos^3(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$- 240 \cos(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$+ 312 \cos^3(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$- 208 \cos(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$+ 480 \cos^3(apk) \sin(fk) \cos(2\, fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$- 320 \cos(apk) \sin(fk) \cos(2\, fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$+ 1080 \cos^3(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$- 720 \cos(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$+ 336 \cos^3(apk)\, ek^2 \sin(fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$- 224 \cos(apk)\, ek^2 \sin(fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$+ 240 \cos^3(apk) \sin(fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$- 160 \cos(apk) \sin(fk) \cos(jk) \cos^2(wk) \sin(wk)$$
$$- 12 \cos^2(apk) \sin(apk)\, ek^2 \cos(5\, fk) \cos^2(wk) \sin(wk)$$
$$- 60 \cos^2(apk) \sin(apk)\, ek \cos(4\, fk) \cos^2(wk) \sin(wk)$$
$$- 40 \cos^2(apk) \sin(apk)\, ek^2 \cos(3\, fk) \cos^2(wk) \sin(wk)$$
$$- 80 \cos^2(apk) \sin(apk) \cos(3\, fk) \cos^2(wk) \sin(wk)$$
$$- 120 \cos^2(apk) \sin(apk)\, ek \cos(2\, fk) \cos^2(wk) \sin(wk)$$
$$- 60 \cos^2(apk) \sin(apk)\, ek^2 \cos(fk) \cos^2(wk) \sin(wk)$$
$$- 6 \cos^3(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^3(jk) \sin(wk)$$
$$+ 6 \cos(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^3(jk) \sin(wk)$$
$$- 30 \cos^3(apk)\, ek \sin(fk) \cos(3\, fk) \cos^3(jk) \sin(wk)$$
$$+ 30 \cos(apk)\, ek \sin(fk) \cos(3\, fk) \cos^3(jk) \sin(wk)$$
$$- 56 \cos^3(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^3(jk) \sin(wk)$$
$$+ 56 \cos(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^3(jk) \sin(wk)$$
$$- 40 \cos^3(apk) \sin(fk) \cos(2\, fk) \cos^3(jk) \sin(wk)$$
$$+ 40 \cos(apk) \sin(fk) \cos(2\, fk) \cos^3(jk) \sin(wk)$$
$$- 270 \cos^3(apk)\, ek \cos(fk) \sin(fk) \cos^3(jk) \sin(wk)$$
$$+ 270 \cos(apk)\, ek \cos(fk) \sin(fk) \cos^3(jk) \sin(wk)$$
$$- 178 \cos^3(apk)\, ek^2 \sin(fk) \cos^3(jk) \sin(wk)$$

$$+ 178 \cos(apk)\, ek^2 \sin(fk) \cos^3(jk) \sin(wk)$$
$$- 200 \cos^3(apk) \sin(fk) \cos^3(jk) \sin(wk)$$
$$+ 200 \cos(apk) \sin(fk) \cos^3(jk) \sin(wk)$$
$$+ 9 \cos^2(apk) \sin(apk)\, ek^2 \cos(5\, fk) \cos^2(jk) \sin(wk)$$
$$- 3 \sin(apk)\, ek^2 \cos(5\, fk) \cos^2(jk) \sin(wk)$$
$$+ 45 \cos^2(apk) \sin(apk)\, ek \cos(4\, fk) \cos^2(jk) \sin(wk)$$
$$- 15 \sin(apk)\, ek \cos(4\, fk) \cos^2(jk) \sin(wk)$$
$$+ 45 \cos^2(apk) \sin(apk)\, ek^2 \cos(3\, fk) \cos^2(jk) \sin(wk)$$
$$- 15 \sin(apk)\, ek^2 \cos(3\, fk) \cos^2(jk) \sin(wk)$$
$$+ 60 \cos^2(apk) \sin(apk) \cos(3\, fk) \cos^2(jk) \sin(wk)$$
$$- 20 \sin(apk) \cos(3\, fk) \cos^2(jk) \sin(wk)$$
$$+ 180 \cos^2(apk) \sin(apk)\, ek \cos(2\, fk) \cos^2(jk) \sin(wk)$$
$$- 60 \sin(apk)\, ek \cos(2\, fk) \cos^2(jk) \sin(wk)$$
$$+ 90 \cos^2(apk) \sin(apk)\, ek^2 \cos(fk) \cos^2(jk) \sin(wk)$$
$$- 30 \sin(apk)\, ek^2 \cos(fk) \cos^2(jk) \sin(wk)$$
$$+ 180 \cos^2(apk) \sin(apk) \cos(fk) \cos^2(jk) \sin(wk)$$
$$- 60 \sin(apk) \cos(fk) \cos^2(jk) \sin(wk)$$
$$- 18 \cos^3(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos(jk) \sin(wk)$$
$$+ 12 \cos(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos(jk) \sin(wk)$$
$$- 90 \cos^3(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \sin(wk)$$
$$+ 60 \cos(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \sin(wk)$$
$$- 48 \cos^3(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos(jk) \sin(wk)$$
$$+ 32 \cos(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos(jk) \sin(wk)$$
$$- 120 \cos^3(apk) \sin(fk) \cos(2\, fk) \cos(jk) \sin(wk)$$
$$+ 80 \cos(apk) \sin(fk) \cos(2\, fk) \cos(jk) \sin(wk)$$
$$- 90 \cos^3(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \sin(wk)$$
$$+ 60 \cos(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \sin(wk)$$
$$+ 66 \cos^3(apk)\, ek^2 \sin(fk) \cos(jk) \sin(wk)$$
$$- 44 \cos(apk)\, ek^2 \sin(fk) \cos(jk) \sin(wk)$$

$$
\begin{aligned}
&+ 120 \cos^{3}(apk) \sin(fk) \cos(jk) \sin(wk) - 80 \cos(apk) \sin(fk) \cos(jk) \sin(wk) \\
&+ 3 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(5\, fk) \sin(wk) \\
&+ 15 \cos^{2}(apk) \sin(apk)\, ek \cos(4\, fk) \sin(wk) \\
&- 5 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(3\, fk) \sin(wk) \\
&+ 20 \cos^{2}(apk) \sin(apk) \cos(3\, fk) \sin(wk) \\
&- 60 \cos^{2}(apk) \sin(apk)\, ek \cos(2\, fk) \sin(wk) \\
&- 30 \cos^{2}(apk) \sin(apk)\, ek^{2} \cos(fk) \sin(wk) \\
&- 180 \cos^{2}(apk) \sin(apk) \cos(fk) \sin(wk) \\
&- 12 \cos^{3}(apk)\, ek^{2} \cos(5\, fk) \cos^{3}(jk) \cos^{3}(wk) \\
&+ 12 \cos(apk)\, ek^{2} \cos(5\, fk) \cos^{3}(jk) \cos^{3}(wk) \\
&- 60 \cos^{3}(apk)\, ek \cos(4\, fk) \cos^{3}(jk) \cos^{3}(wk) \\
&+ 60 \cos(apk)\, ek \cos(4\, fk) \cos^{3}(jk) \cos^{3}(wk) \\
&- 40 \cos^{3}(apk)\, ek^{2} \cos(3\, fk) \cos^{3}(jk) \cos^{3}(wk) \\
&+ 40 \cos(apk)\, ek^{2} \cos(3\, fk) \cos^{3}(jk) \cos^{3}(wk) \\
&- 80 \cos^{3}(apk) \cos(3\, fk) \cos^{3}(jk) \cos^{3}(wk) \\
&+ 80 \cos(apk) \cos(3\, fk) \cos^{3}(jk) \cos^{3}(wk) \\
&- 120 \cos^{3}(apk)\, ek \cos(2\, fk) \cos^{3}(jk) \cos^{3}(wk) \\
&+ 120 \cos(apk)\, ek \cos(2\, fk) \cos^{3}(jk) \cos^{3}(wk) \\
&- 60 \cos^{3}(apk)\, ek^{2} \cos(fk) \cos^{3}(jk) \cos^{3}(wk) \\
&+ 60 \cos(apk)\, ek^{2} \cos(fk) \cos^{3}(jk) \cos^{3}(wk) \\
&- 72 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&+ 24 \sin(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&- 360 \cos^{2}(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&+ 120 \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&- 312 \cos^{2}(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&+ 104 \sin(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&- 480 \cos^{2}(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&+ 160 \sin(apk) \sin(fk) \cos(2\, fk) \cos^{2}(jk) \cos^{3}(wk) \\
&- 1080 \cos^{2}(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos^{2}(jk) \cos^{3}(wk)
\end{aligned}
$$

$$+ 360 \sin(apk)\, ek \cos(fk) \sin(fk) \cos^2(jk) \cos^3(wk)$$
$$- 336 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos^2(jk) \cos^3(wk)$$
$$+ 112 \sin(apk)\, ek^2 \sin(fk) \cos^2(jk) \cos^3(wk)$$
$$- 240 \cos^2(apk) \sin(apk) \sin(fk) \cos^2(jk) \cos^3(wk)$$
$$+ 80 \sin(apk) \sin(fk) \cos^2(jk) \cos^3(wk)$$
$$- 36 \cos^3(apk)\, ek^2 \cos(5\, fk) \cos(jk) \cos^3(wk)$$
$$+ 24 \cos(apk)\, ek^2 \cos(5\, fk) \cos(jk) \cos^3(wk)$$
$$- 180 \cos^3(apk)\, ek \cos(4\, fk) \cos(jk) \cos^3(wk)$$
$$+ 120 \cos(apk)\, ek \cos(4\, fk) \cos(jk) \cos^3(wk)$$
$$- 120 \cos^3(apk)\, ek^2 \cos(3\, fk) \cos(jk) \cos^3(wk)$$
$$+ 80 \cos(apk)\, ek^2 \cos(3\, fk) \cos(jk) \cos^3(wk)$$
$$- 240 \cos^3(apk) \cos(3\, fk) \cos(jk) \cos^3(wk)$$
$$+ 160 \cos(apk) \cos(3\, fk) \cos(jk) \cos^3(wk)$$
$$- 360 \cos^3(apk)\, ek \cos(2\, fk) \cos(jk) \cos^3(wk)$$
$$+ 240 \cos(apk)\, ek \cos(2\, fk) \cos(jk) \cos^3(wk)$$
$$- 180 \cos^3(apk)\, ek^2 \cos(fk) \cos(jk) \cos^3(wk)$$
$$+ 120 \cos(apk)\, ek^2 \cos(fk) \cos(jk) \cos^3(wk)$$
$$- 24 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^3(wk)$$
$$- 120 \cos^2(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^3(wk)$$
$$- 104 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^3(wk)$$
$$- 160 \cos^2(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos^3(wk)$$
$$- 360 \cos^2(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos^3(wk)$$
$$- 112 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos^3(wk)$$
$$- 80 \cos^2(apk) \sin(apk) \sin(fk) \cos^3(wk)$$
$$+ 9 \cos^3(apk)\, ek^2 \cos(5\, fk) \cos^3(jk) \cos(wk)$$
$$- 9 \cos(apk)\, ek^2 \cos(5\, fk) \cos^3(jk) \cos(wk)$$
$$+ 45 \cos^3(apk)\, ek \cos(4\, fk) \cos^3(jk) \cos(wk)$$
$$- 45 \cos(apk)\, ek \cos(4\, fk) \cos^3(jk) \cos(wk)$$
$$+ 45 \cos^3(apk)\, ek^2 \cos(3\, fk) \cos^3(jk) \cos(wk)$$

$$- 45 \cos(apk)\, ek^2 \cos(3\, fk) \cos^3(jk) \cos(wk)$$

$$+ 60 \cos^3(apk) \cos(3\, fk) \cos^3(jk) \cos(wk)$$

$$- 60 \cos(apk) \cos(3\, fk) \cos^3(jk) \cos(wk)$$

$$+ 180 \cos^3(apk)\, ek \cos(2\, fk) \cos^3(jk) \cos(wk)$$

$$- 180 \cos(apk)\, ek \cos(2\, fk) \cos^3(jk) \cos(wk)$$

$$+ 90 \cos^3(apk)\, ek^2 \cos(fk) \cos^3(jk) \cos(wk)$$

$$- 90 \cos(apk)\, ek^2 \cos(fk) \cos^3(jk) \cos(wk)$$

$$+ 180 \cos^3(apk) \cos(fk) \cos^3(jk) \cos(wk)$$

$$- 180 \cos(apk) \cos(fk) \cos^3(jk) \cos(wk)$$

$$+ 54 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(jk) \cos(wk)$$

$$- 18 \sin(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(jk) \cos(wk)$$

$$+ 270 \cos^2(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^2(jk) \cos(wk)$$

$$- 90 \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^2(jk) \cos(wk)$$

$$+ 264 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(jk) \cos(wk)$$

$$- 88 \sin(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(jk) \cos(wk)$$

$$+ 360 \cos^2(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos^2(jk) \cos(wk)$$

$$- 120 \sin(apk) \sin(fk) \cos(2\, fk) \cos^2(jk) \cos(wk)$$

$$+ 990 \cos^2(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos^2(jk) \cos(wk)$$

$$- 330 \sin(apk)\, ek \cos(fk) \sin(fk) \cos^2(jk) \cos(wk)$$

$$+ 402 \cos^2(apk) \sin(apk)\, ek^2 \sin(fk) \cos^2(jk) \cos(wk)$$

$$- 134 \sin(apk)\, ek^2 \sin(fk) \cos^2(jk) \cos(wk)$$

$$+ 360 \cos^2(apk) \sin(apk) \sin(fk) \cos^2(jk) \cos(wk)$$

$$- 120 \sin(apk) \sin(fk) \cos^2(jk) \cos(wk)$$

$$+ 27 \cos^3(apk)\, ek^2 \cos(5\, fk) \cos(jk) \cos(wk)$$

$$- 18 \cos(apk)\, ek^2 \cos(5\, fk) \cos(jk) \cos(wk)$$

$$+ 135 \cos^3(apk)\, ek \cos(4\, fk) \cos(jk) \cos(wk)$$

$$- 90 \cos(apk)\, ek \cos(4\, fk) \cos(jk) \cos(wk)$$

$$+ 75 \cos^3(apk)\, ek^2 \cos(3\, fk) \cos(jk) \cos(wk)$$

$$- 50 \cos(apk)\, ek^2 \cos(3\, fk) \cos(jk) \cos(wk)$$

```
         3
+ 180 cos (apk) cos(3 fk) cos(jk) cos(wk)

- 120 cos(apk) cos(3 fk) cos(jk) cos(wk)
         3
+ 180 cos (apk) ek cos(2 fk) cos(jk) cos(wk)

- 120 cos(apk) ek cos(2 fk) cos(jk) cos(wk)
        3        2
+ 90 cos (apk) ek  cos(fk) cos(jk) cos(wk)
                2
- 60 cos(apk) ek  cos(fk) cos(jk) cos(wk)
         3
- 180 cos (apk) cos(fk) cos(jk) cos(wk) + 120 cos(apk) cos(fk) cos(jk) cos(wk)
        2                 2
+ 18 cos (apk) sin(apk) ek  sin(fk) cos(4 fk) cos(wk)
        2
+ 90 cos (apk) sin(apk) ek sin(fk) cos(3 fk) cos(wk)
        2                 2
+ 48 cos (apk) sin(apk) ek  sin(fk) cos(2 fk) cos(wk)
         2
+ 120 cos (apk) sin(apk) sin(fk) cos(2 fk) cos(wk)
        2
+ 90 cos (apk) sin(apk) ek cos(fk) sin(fk) cos(wk)
        2                 2
- 66 cos (apk) sin(apk) ek  sin(fk) cos(wk)
         2
- 120 cos (apk) sin(apk) sin(fk) cos(wk)) zb2/(240 nk)
```

$\delta(za2S1S2S3) =$

```
                                         2                      2        2
     - sin(jk) n (24 cos(apk) sin(apk) ek  sin(fk) cos(4 fk) cos (jk) cos (wk)
                                                        2        2
sin(wk) + 120 cos(apk) sin(apk) ek sin(fk) cos(3 fk) cos (jk) cos (wk) sin(wk)
                          2                      2        2
+ 104 cos(apk) sin(apk) ek  sin(fk) cos(2 fk) cos (jk) cos (wk) sin(wk)
                                             2        2
+ 160 cos(apk) sin(apk) sin(fk) cos(2 fk) cos (jk) cos (wk) sin(wk)
                                              2        2
+ 360 cos(apk) sin(apk) ek cos(fk) sin(fk) cos (jk) cos (wk) sin(wk)
                          2            2        2
+ 112 cos(apk) sin(apk) ek  sin(fk) cos (jk) cos (wk) sin(wk)
                                  2        2
+ 80 cos(apk) sin(apk) sin(fk) cos (jk) cos (wk) sin(wk)
        2        2                      2
+ 24 cos (apk) ek  cos(5 fk) cos(jk) cos (wk) sin(wk)
       2                      2
- 12 ek  cos(5 fk) cos(jk) cos (wk) sin(wk)
         2                              2
+ 120 cos (apk) ek cos(4 fk) cos(jk) cos (wk) sin(wk)
                             2
- 60 ek cos(4 fk) cos(jk) cos (wk) sin(wk)
        2        2                      2
+ 80 cos (apk) ek  cos(3 fk) cos(jk) cos (wk) sin(wk)
```

$$- 40\, ek^{2} \cos(3\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$
$$+ 160 \cos^{2}(apk) \cos(3\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$
$$- 80 \cos(3\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$
$$+ 240 \cos^{2}(apk)\, ek \cos(2\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$
$$- 120\, ek \cos(2\, fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$
$$+ 120 \cos^{2}(apk)\, ek^{2} \cos(fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$
$$- 60\, ek^{2} \cos(fk) \cos(jk) \cos^{2}(wk) \sin(wk)$$
$$+ 24 \cos(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos^{2}(wk) \sin(wk)$$
$$+ 120 \cos(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^{2}(wk) \sin(wk)$$
$$+ 104 \cos(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos^{2}(wk) \sin(wk)$$
$$+ 160 \cos(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos^{2}(wk) \sin(wk)$$
$$+ 360 \cos(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos^{2}(wk) \sin(wk)$$
$$+ 112 \cos(apk) \sin(apk)\, ek^{2} \sin(fk) \cos^{2}(wk) \sin(wk)$$
$$+ 80 \cos(apk) \sin(apk) \sin(fk) \cos^{2}(wk) \sin(wk)$$
$$- 6 \cos(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos^{2}(jk) \sin(wk)$$
$$- 30 \cos(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos^{2}(jk) \sin(wk)$$
$$- 56 \cos(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos^{2}(jk) \sin(wk)$$
$$- 40 \cos(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos^{2}(jk) \sin(wk)$$
$$- 270 \cos(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos^{2}(jk) \sin(wk)$$
$$- 178 \cos(apk) \sin(apk)\, ek^{2} \sin(fk) \cos^{2}(jk) \sin(wk)$$
$$- 200 \cos(apk) \sin(apk) \sin(fk) \cos^{2}(jk) \sin(wk)$$
$$- 6 \cos^{2}(apk)\, ek^{2} \cos(5\, fk) \cos(jk) \sin(wk) + 3\, ek^{2} \cos(5\, fk) \cos(jk) \sin(wk)$$
$$- 30 \cos^{2}(apk)\, ek \cos(4\, fk) \cos(jk) \sin(wk) + 15\, ek \cos(4\, fk) \cos(jk) \sin(wk)$$
$$- 30 \cos^{2}(apk)\, ek^{2} \cos(3\, fk) \cos(jk) \sin(wk)$$
$$+ 15\, ek^{2} \cos(3\, fk) \cos(jk) \sin(wk) - 40 \cos^{2}(apk) \cos(3\, fk) \cos(jk) \sin(wk)$$
$$+ 20 \cos(3\, fk) \cos(jk) \sin(wk) - 120 \cos^{2}(apk)\, ek \cos(2\, fk) \cos(jk) \sin(wk)$$
$$+ 60\, ek \cos(2\, fk) \cos(jk) \sin(wk) - 60 \cos^{2}(apk)\, ek^{2} \cos(fk) \cos(jk) \sin(wk)$$
$$+ 30\, ek^{2} \cos(fk) \cos(jk) \sin(wk) - 120 \cos^{2}(apk) \cos(fk) \cos(jk) \sin(wk)$$
$$+ 60 \cos(fk) \cos(jk) \sin(wk) - 6 \cos(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(4\, fk)$$

$$\sin(wk) - 30 \cos(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \sin(wk)$$

$$- 16 \cos(apk) \sin(apk)\, ek^2 \sin(fk) \cos(2\, fk) \sin(wk)$$

$$- 40 \cos(apk) \sin(apk) \sin(fk) \cos(2\, fk) \sin(wk)$$

$$- 30 \cos(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \sin(wk)$$

$$+ 22 \cos(apk) \sin(apk)\, ek^2 \sin(fk) \sin(wk)$$

$$+ 40 \cos(apk) \sin(apk) \sin(fk) \sin(wk)$$

$$- 12 \cos(apk) \sin(apk)\, ek^2 \cos(5\, fk) \cos^2(jk) \cos^3(wk)$$

$$- 60 \cos(apk) \sin(apk)\, ek \cos(4\, fk) \cos^2(jk) \cos^3(wk)$$

$$- 40 \cos(apk) \sin(apk)\, ek^2 \cos(3\, fk) \cos^2(jk) \cos^3(wk)$$

$$- 80 \cos(apk) \sin(apk) \cos(3\, fk) \cos^2(jk) \cos^3(wk)$$

$$- 120 \cos(apk) \sin(apk)\, ek \cos(2\, fk) \cos^2(jk) \cos^3(wk)$$

$$- 60 \cos(apk) \sin(apk)\, ek^2 \cos(fk) \cos^2(jk) \cos^3(wk)$$

$$+ 48 \cos^2(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos(jk) \cos^3(wk)$$

$$- 24\, ek^2 \sin(fk) \cos(4\, fk) \cos(jk) \cos^3(wk)$$

$$+ 240 \cos^2(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \cos^3(wk)$$

$$- 120\, ek \sin(fk) \cos(3\, fk) \cos(jk) \cos^3(wk)$$

$$+ 208 \cos^2(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos(jk) \cos^3(wk)$$

$$- 104\, ek^2 \sin(fk) \cos(2\, fk) \cos(jk) \cos^3(wk)$$

$$+ 320 \cos^2(apk) \sin(fk) \cos(2\, fk) \cos(jk) \cos^3(wk)$$

$$- 160 \sin(fk) \cos(2\, fk) \cos(jk) \cos^3(wk)$$

$$+ 720 \cos^2(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \cos^3(wk)$$

$$- 360\, ek \cos(fk) \sin(fk) \cos(jk) \cos^3(wk)$$

$$+ 224 \cos^2(apk)\, ek^2 \sin(fk) \cos(jk) \cos^3(wk)$$

$$- 112\, ek^2 \sin(fk) \cos(jk) \cos^3(wk) + 160 \cos^2(apk) \sin(fk) \cos(jk) \cos^3(wk)$$

$$- 80 \sin(fk) \cos(jk) \cos^3(wk) - 12 \cos(apk) \sin(apk)\, ek^2 \cos(5\, fk) \cos^3(wk)$$

$$- 60 \cos(apk) \sin(apk)\, ek \cos(4\, fk) \cos^3(wk)$$

$$- 40 \cos(apk) \sin(apk)\, ek^2 \cos(3\, fk) \cos^3(wk)$$

$$- 80 \cos(apk) \sin(apk) \cos(3\, fk) \cos^3(wk)$$

$$- 120 \cos(apk) \sin(apk)\, ek \cos(2\, fk) \cos^3(wk)$$

$$- 60 \cos(apk) \sin(apk)\, ek^2 \cos(fk) \cos^3(wk)$$

$$
\begin{aligned}
&+ 9 \cos(apk) \sin(apk)\, ek^{2} \cos(5\, fk) \cos^{2}(jk) \cos(wk) \\
&+ 45 \cos(apk) \sin(apk)\, ek \cos(4\, fk) \cos^{2}(jk) \cos(wk) \\
&+ 45 \cos(apk) \sin(apk)\, ek^{2} \cos(3\, fk) \cos^{2}(jk) \cos(wk) \\
&+ 60 \cos(apk) \sin(apk) \cos(3\, fk) \cos^{2}(jk) \cos(wk) \\
&+ 180 \cos(apk) \sin(apk)\, ek \cos(2\, fk) \cos^{2}(jk) \cos(wk) \\
&+ 90 \cos(apk) \sin(apk)\, ek^{2} \cos(fk) \cos^{2}(jk) \cos(wk) \\
&+ 180 \cos(apk) \sin(apk) \cos(fk) \cos^{2}(jk) \cos(wk) \\
&- 36 \cos^{2}(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos(jk) \cos(wk) \\
&+ 18\, ek^{2} \sin(fk) \cos(4\, fk) \cos(jk) \cos(wk) \\
&- 180 \cos^{2}(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \cos(wk) \\
&+ 90\, ek \sin(fk) \cos(3\, fk) \cos(jk) \cos(wk) \\
&- 176 \cos^{2}(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos(jk) \cos(wk) \\
&+ 88\, ek^{2} \sin(fk) \cos(2\, fk) \cos(jk) \cos(wk) \\
&- 240 \cos^{2}(apk) \sin(fk) \cos(2\, fk) \cos(jk) \cos(wk) \\
&+ 120 \sin(fk) \cos(2\, fk) \cos(jk) \cos(wk) \\
&- 660 \cos^{2}(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \cos(wk) \\
&+ 330\, ek \cos(fk) \sin(fk) \cos(jk) \cos(wk) \\
&- 268 \cos^{2}(apk)\, ek^{2} \sin(fk) \cos(jk) \cos(wk) + 134\, ek^{2} \sin(fk) \cos(jk) \cos(wk) \\
&- 240 \cos^{2}(apk) \sin(fk) \cos(jk) \cos(wk) + 120 \sin(fk) \cos(jk) \cos(wk) \\
&+ 9 \cos(apk) \sin(apk)\, ek^{2} \cos(5\, fk) \cos(wk) \\
&+ 45 \cos(apk) \sin(apk)\, ek \cos(4\, fk) \cos(wk) \\
&+ 25 \cos(apk) \sin(apk)\, ek^{2} \cos(3\, fk) \cos(wk) \\
&+ 60 \cos(apk) \sin(apk) \cos(3\, fk) \cos(wk) \\
&+ 60 \cos(apk) \sin(apk)\, ek \cos(2\, fk) \cos(wk) \\
&+ 30 \cos(apk) \sin(apk)\, ek^{2} \cos(fk) \cos(wk) \\
&- 60 \cos(apk) \sin(apk) \cos(fk) \cos(wk))\, zb2/(240\, nk)
\end{aligned}
$$

$\delta(za2S3S2^2) =$

$$
\begin{aligned}
&- \sin(jk)\, n\, (24 \cos^2(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 120 \cos^2(apk)\, ek \sin(fk) \cos(3\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 104 \cos^2(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 160 \cos^2(apk) \sin(fk) \cos(2\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 360 \cos^2(apk)\, ek \cos(fk) \sin(fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 112 \cos^2(apk)\, ek^2 \sin(fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&+ 80 \cos^2(apk) \sin(fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
&- 24 \cos(apk) \sin(apk)\, ek^2 \cos(5\, fk) \cos(jk) \cos^2(wk) \sin(wk) \\
&- 120 \cos(apk) \sin(apk)\, ek \cos(4\, fk) \cos(jk) \cos^2(wk) \sin(wk) \\
&- 80 \cos(apk) \sin(apk)\, ek^2 \cos(3\, fk) \cos(jk) \cos^2(wk) \sin(wk) \\
&- 160 \cos(apk) \sin(apk) \cos(3\, fk) \cos(jk) \cos^2(wk) \sin(wk) \\
&- 240 \cos(apk) \sin(apk)\, ek \cos(2\, fk) \cos(jk) \cos^2(wk) \sin(wk) \\
&- 120 \cos(apk) \sin(apk)\, ek^2 \cos(fk) \cos(jk) \cos^2(wk) \sin(wk) \\
&+ 24 \cos^2(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(wk) \sin(wk) \\
&- 24\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(wk) \sin(wk) \\
&+ 120 \cos^2(apk)\, ek \sin(fk) \cos(3\, fk) \cos^2(wk) \sin(wk) \\
&- 120\, ek \sin(fk) \cos(3\, fk) \cos^2(wk) \sin(wk) \\
&+ 104 \cos^2(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(wk) \sin(wk) \\
&- 104\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(wk) \sin(wk) \\
&+ 160 \cos^2(apk) \sin(fk) \cos(2\, fk) \cos^2(wk) \sin(wk) \\
&- 160 \sin(fk) \cos(2\, fk) \cos^2(wk) \sin(wk) \\
&+ 360 \cos^2(apk)\, ek \cos(fk) \sin(fk) \cos^2(wk) \sin(wk) \\
&- 360\, ek \cos(fk) \sin(fk) \cos^2(wk) \sin(wk) \\
&+ 112 \cos^2(apk)\, ek^2 \sin(fk) \cos^2(wk) \sin(wk) \\
&- 112\, ek^2 \sin(fk) \cos^2(wk) \sin(wk) + 80 \cos^2(apk) \sin(fk) \cos^2(wk) \sin(wk) \\
&- 80 \sin(fk) \cos^2(wk) \sin(wk) - 6 \cos^2(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(jk) \\
&\sin(wk) - 30 \cos^2(apk)\, ek \sin(fk) \cos(3\, fk) \cos^2(jk) \sin(wk) \\
&- 56 \cos^2(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(jk) \sin(wk)
\end{aligned}
$$

$$- 40 \cos^2(apk) \sin(fk) \cos(2 fk) \cos^2(jk) \sin(wk)$$
$$- 270 \cos^2(apk)\, ek \cos(fk) \sin(fk) \cos^2(jk) \sin(wk)$$
$$- 178 \cos^2(apk)\, ek^2 \sin(fk) \cos^2(jk) \sin(wk)$$
$$- 200 \cos^2(apk) \sin(fk) \cos^2(jk) \sin(wk)$$
$$+ 6 \cos(apk) \sin(apk)\, ek^2 \cos(5 fk) \cos(jk) \sin(wk)$$
$$+ 30 \cos(apk) \sin(apk)\, ek \cos(4 fk) \cos(jk) \sin(wk)$$
$$+ 30 \cos(apk) \sin(apk)\, ek^2 \cos(3 fk) \cos(jk) \sin(wk)$$
$$+ 40 \cos(apk) \sin(apk) \cos(3 fk) \cos(jk) \sin(wk)$$
$$+ 120 \cos(apk) \sin(apk)\, ek \cos(2 fk) \cos(jk) \sin(wk)$$
$$+ 60 \cos(apk) \sin(apk)\, ek^2 \cos(fk) \cos(jk) \sin(wk)$$
$$+ 120 \cos(apk) \sin(apk) \cos(fk) \cos(jk) \sin(wk)$$
$$- 6 \cos^2(apk)\, ek^2 \sin(fk) \cos(4 fk) \sin(wk) + 6\, ek^2 \sin(fk) \cos(4 fk) \sin(wk)$$
$$- 30 \cos^2(apk)\, ek \sin(fk) \cos(3 fk) \sin(wk) + 30\, ek \sin(fk) \cos(3 fk) \sin(wk)$$
$$- 16 \cos^2(apk)\, ek^2 \sin(fk) \cos(2 fk) \sin(wk)$$
$$+ 16\, ek^2 \sin(fk) \cos(2 fk) \sin(wk) - 40 \cos^2(apk) \sin(fk) \cos(2 fk) \sin(wk)$$
$$+ 40 \sin(fk) \cos(2 fk) \sin(wk) - 30 \cos^2(apk)\, ek \cos(fk) \sin(fk) \sin(wk)$$
$$+ 30\, ek \cos(fk) \sin(fk) \sin(wk) + 22 \cos^2(apk)\, ek^2 \sin(fk) \sin(wk)$$
$$- 22\, ek^2 \sin(fk) \sin(wk) + 40 \cos^2(apk) \sin(fk) \sin(wk) - 40 \sin(fk) \sin(wk)$$
$$- 12 \cos^2(apk)\, ek^2 \cos(5 fk) \cos^2(jk) \cos^3(wk)$$
$$- 60 \cos^2(apk)\, ek \cos(4 fk) \cos^2(jk) \cos^3(wk)$$
$$- 40 \cos^2(apk)\, ek^2 \cos(3 fk) \cos^2(jk) \cos^3(wk)$$
$$- 80 \cos^2(apk) \cos(3 fk) \cos^2(jk) \cos^3(wk)$$
$$- 120 \cos^2(apk)\, ek \cos(2 fk) \cos^2(jk) \cos^3(wk)$$
$$- 60 \cos^2(apk)\, ek^2 \cos(fk) \cos^2(jk) \cos^3(wk)$$
$$- 48 \cos(apk) \sin(apk)\, ek^2 \sin(fk) \cos(4 fk) \cos(jk) \cos^3(wk)$$
$$- 240 \cos(apk) \sin(apk)\, ek \sin(fk) \cos(3 fk) \cos(jk) \cos^3(wk)$$
$$- 208 \cos(apk) \sin(apk)\, ek^2 \sin(fk) \cos(2 fk) \cos(jk) \cos^3(wk)$$
$$- 320 \cos(apk) \sin(apk) \sin(fk) \cos(2 fk) \cos(jk) \cos^3(wk)$$
$$- 720 \cos(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \cos^3(wk)$$

$$- 224 \cos(apk) \sin(apk) ek^{2} \sin(fk) \cos(jk) \cos^{3}(wk)$$
$$- 160 \cos(apk) \sin(apk) \sin(fk) \cos(jk) \cos^{3}(wk)$$
$$- 12 \cos^{2}(apk) ek^{2} \cos(5 fk) \cos^{3}(wk) + 12 ek^{2} \cos(5 fk) \cos^{3}(wk)$$
$$- 60 \cos^{2}(apk) ek \cos(4 fk) \cos^{3}(wk) + 60 ek \cos(4 fk) \cos^{3}(wk)$$
$$- 40 \cos^{2}(apk) ek^{2} \cos(3 fk) \cos^{3}(wk) + 40 ek^{2} \cos(3 fk) \cos^{3}(wk)$$
$$- 80 \cos^{2}(apk) \cos(3 fk) \cos^{3}(wk) + 80 \cos(3 fk) \cos^{3}(wk)$$
$$- 120 \cos^{2}(apk) ek \cos(2 fk) \cos^{3}(wk) + 120 ek \cos(2 fk) \cos^{3}(wk)$$
$$- 60 \cos^{2}(apk) ek^{2} \cos(fk) \cos^{3}(wk) + 60 ek^{2} \cos(fk) \cos^{3}(wk)$$
$$+ 9 \cos^{2}(apk) ek^{2} \cos(5 fk) \cos^{2}(jk) \cos(wk)$$
$$+ 45 \cos^{2}(apk) ek \cos(4 fk) \cos^{2}(jk) \cos(wk)$$
$$+ 45 \cos^{2}(apk) ek^{2} \cos(3 fk) \cos^{2}(jk) \cos(wk)$$
$$+ 60 \cos^{2}(apk) \cos(3 fk) \cos^{2}(jk) \cos(wk)$$
$$+ 180 \cos^{2}(apk) ek \cos(2 fk) \cos^{2}(jk) \cos(wk)$$
$$+ 90 \cos^{2}(apk) ek^{2} \cos(fk) \cos^{2}(jk) \cos(wk)$$
$$+ 180 \cos^{2}(apk) \cos(fk) \cos^{2}(jk) \cos(wk)$$
$$+ 36 \cos(apk) \sin(apk) ek^{2} \sin(fk) \cos(4 fk) \cos(jk) \cos(wk)$$
$$+ 180 \cos(apk) \sin(apk) ek \sin(fk) \cos(3 fk) \cos(jk) \cos(wk)$$
$$+ 176 \cos(apk) \sin(apk) ek^{2} \sin(fk) \cos(2 fk) \cos(jk) \cos(wk)$$
$$+ 240 \cos(apk) \sin(apk) \sin(fk) \cos(2 fk) \cos(jk) \cos(wk)$$
$$+ 660 \cos(apk) \sin(apk) ek \cos(fk) \sin(fk) \cos(jk) \cos(wk)$$
$$+ 268 \cos(apk) \sin(apk) ek^{2} \sin(fk) \cos(jk) \cos(wk)$$
$$+ 240 \cos(apk) \sin(apk) \sin(fk) \cos(jk) \cos(wk)$$
$$+ 9 \cos^{2}(apk) ek^{2} \cos(5 fk) \cos(wk) - 9 ek^{2} \cos(5 fk) \cos(wk)$$
$$+ 45 \cos^{2}(apk) ek \cos(4 fk) \cos(wk) - 45 ek \cos(4 fk) \cos(wk)$$
$$+ 25 \cos^{2}(apk) ek^{2} \cos(3 fk) \cos(wk) - 25 ek^{2} \cos(3 fk) \cos(wk)$$
$$+ 60 \cos^{2}(apk) \cos(3 fk) \cos(wk) - 60 \cos(3 fk) \cos(wk)$$
$$+ 60 \cos^{2}(apk) ek \cos(2 fk) \cos(wk) - 60 ek \cos(2 fk) \cos(wk)$$
$$+ 30 \cos^{2}(apk) ek^{2} \cos(fk) \cos(wk) - 30 ek^{2} \cos(fk) \cos(wk)$$
$$- 60 \cos^{2}(apk) \cos(fk) \cos(wk) + 60 \cos(fk) \cos(wk)) zb2/(240 nk)$$

$\delta(za2S3S1^2) =$

$$
\begin{aligned}
& \sin(jk)\, n\, (24 \cos^2(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
& - 24\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
& + 120 \cos^2(apk)\, ek \sin(fk) \cos(3\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
& - 120\, ek \sin(fk) \cos(3\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
& + 104 \cos^2(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
& - 104\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
& + 160 \cos^2(apk) \sin(fk) \cos(2\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
& - 160 \sin(fk) \cos(2\, fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
& + 360 \cos^2(apk)\, ek \cos(fk) \sin(fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
& - 360\, ek \cos(fk) \sin(fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
& + 112 \cos^2(apk)\, ek^2 \sin(fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
& - 112\, ek^2 \sin(fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
& + 80 \cos^2(apk) \sin(fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
& - 80 \sin(fk) \cos^2(jk) \cos^2(wk) \sin(wk) \\
& - 24 \cos(apk) \sin(apk)\, ek^2 \cos(5\, fk) \cos(jk) \cos^2(wk) \sin(wk) \\
& - 120 \cos(apk) \sin(apk)\, ek \cos(4\, fk) \cos(jk) \cos^2(wk) \sin(wk) \\
& - 80 \cos(apk) \sin(apk)\, ek^2 \cos(3\, fk) \cos(jk) \cos^2(wk) \sin(wk) \\
& - 160 \cos(apk) \sin(apk) \cos(3\, fk) \cos(jk) \cos^2(wk) \sin(wk) \\
& - 240 \cos(apk) \sin(apk)\, ek \cos(2\, fk) \cos(jk) \cos^2(wk) \sin(wk) \\
& - 120 \cos(apk) \sin(apk)\, ek^2 \cos(fk) \cos(jk) \cos^2(wk) \sin(wk) \\
& + 24 \cos^2(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(wk) \sin(wk) \\
& + 120 \cos^2(apk)\, ek \sin(fk) \cos(3\, fk) \cos^2(wk) \sin(wk) \\
& + 104 \cos^2(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(wk) \sin(wk) \\
& + 160 \cos^2(apk) \sin(fk) \cos(2\, fk) \cos^2(wk) \sin(wk) \\
& + 360 \cos^2(apk)\, ek \cos(fk) \sin(fk) \cos^2(wk) \sin(wk) \\
& + 112 \cos^2(apk)\, ek^2 \sin(fk) \cos^2(wk) \sin(wk) \\
& + 80 \cos^2(apk) \sin(fk) \cos^2(wk) \sin(wk) \\
& - 6 \cos^2(apk)\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(jk) \sin(wk)
\end{aligned}
$$

$$
\begin{aligned}
&+ 6\, ek^2 \sin(fk) \cos(4\, fk) \cos^2(jk) \sin(wk)\\
&- 30 \cos^2(apk)\, ek \sin(fk) \cos(3\, fk) \cos^2(jk) \sin(wk)\\
&+ 30\, ek \sin(fk) \cos(3\, fk) \cos^2(jk) \sin(wk)\\
&- 56 \cos^2(apk)\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(jk) \sin(wk)\\
&+ 56\, ek^2 \sin(fk) \cos(2\, fk) \cos^2(jk) \sin(wk)\\
&- 40 \cos^2(apk) \sin(fk) \cos(2\, fk) \cos^2(jk) \sin(wk)\\
&+ 40 \sin(fk) \cos(2\, fk) \cos^2(jk) \sin(wk)\\
&- 270 \cos^2(apk)\, ek \cos(fk) \sin(fk) \cos^2(jk) \sin(wk)\\
&+ 270\, ek \cos(fk) \sin(fk) \cos^2(jk) \sin(wk)\\
&- 178 \cos^2(apk)\, ek^2 \sin(fk) \cos^2(jk) \sin(wk)\\
&+ 178\, ek^2 \sin(fk) \cos^2(jk) \sin(wk) - 200 \cos^2(apk) \sin(fk) \cos^2(jk) \sin(wk)\\
&+ 200 \sin(fk) \cos^2(jk) \sin(wk) + 6 \cos(apk) \sin(apk)\, ek^2 \cos(5\, fk) \cos(jk)\\
&\sin(wk) + 30 \cos(apk) \sin(apk)\, ek \cos(4\, fk) \cos(jk) \sin(wk)\\
&+ 30 \cos(apk) \sin(apk)\, ek^2 \cos(3\, fk) \cos(jk) \sin(wk)\\
&+ 40 \cos(apk) \sin(apk) \cos(3\, fk) \cos(jk) \sin(wk)\\
&+ 120 \cos(apk) \sin(apk)\, ek \cos(2\, fk) \cos(jk) \sin(wk)\\
&+ 60 \cos(apk) \sin(apk)\, ek^2 \cos(fk) \cos(jk) \sin(wk)\\
&+ 120 \cos(apk) \sin(apk) \cos(fk) \cos(jk) \sin(wk)\\
&- 6 \cos^2(apk)\, ek^2 \sin(fk) \cos(4\, fk) \sin(wk)\\
&- 30 \cos^2(apk)\, ek \sin(fk) \cos(3\, fk) \sin(wk)\\
&- 16 \cos^2(apk)\, ek^2 \sin(fk) \cos(2\, fk) \sin(wk)\\
&- 40 \cos^2(apk) \sin(fk) \cos(2\, fk) \sin(wk)\\
&- 30 \cos^2(apk)\, ek \cos(fk) \sin(fk) \sin(wk) + 22 \cos^2(apk)\, ek^2 \sin(fk) \sin(wk)\\
&+ 40 \cos^2(apk) \sin(fk) \sin(wk) - 12 \cos^2(apk)\, ek^2 \cos(5\, fk) \cos^2(jk) \cos^3(wk)\\
&+ 12\, ek^2 \cos(5\, fk) \cos^2(jk) \cos^3(wk) - 60 \cos^2(apk)\, ek \cos(4\, fk) \cos^2(jk)\\
&\cos^3(wk) + 60\, ek \cos(4\, fk) \cos^2(jk) \cos^3(wk)\\
&- 40 \cos^2(apk)\, ek^2 \cos(3\, fk) \cos^2(jk) \cos^3(wk)\\
&+ 40\, ek^2 \cos(3\, fk) \cos^2(jk) \cos^3(wk) - 80 \cos^2(apk) \cos(3\, fk) \cos^2(jk)\\
&\cos^3(wk) + 80 \cos(3\, fk) \cos^2(jk) \cos^3(wk)
\end{aligned}
$$

$$- 120 \cos^{2}(apk)\, ek \cos(2\, fk) \cos^{2}(jk) \cos^{3}(wk)$$
$$+ 120\, ek \cos(2\, fk) \cos^{2}(jk) \cos^{3}(wk) - 60 \cos^{2}(apk)\, ek^{2} \cos(fk) \cos^{2}(jk)$$
$$\cos^{3}(wk) + 60\, ek^{2} \cos(fk) \cos^{2}(jk) \cos^{3}(wk)$$
$$- 48 \cos(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos(jk) \cos^{3}(wk)$$
$$- 240 \cos(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \cos^{3}(wk)$$
$$- 208 \cos(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos(jk) \cos^{3}(wk)$$
$$- 320 \cos(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos(jk) \cos^{3}(wk)$$
$$- 720 \cos(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \cos^{3}(wk)$$
$$- 224 \cos(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(jk) \cos^{3}(wk)$$
$$- 160 \cos(apk) \sin(apk) \sin(fk) \cos(jk) \cos^{3}(wk)$$
$$- 12 \cos^{2}(apk)\, ek^{2} \cos(5\, fk) \cos^{3}(wk) - 60 \cos^{2}(apk)\, ek \cos(4\, fk) \cos^{3}(wk)$$
$$- 40 \cos^{2}(apk)\, ek^{2} \cos(3\, fk) \cos^{3}(wk) - 80 \cos^{2}(apk) \cos(3\, fk) \cos^{3}(wk)$$
$$- 120 \cos^{2}(apk)\, ek \cos(2\, fk) \cos^{3}(wk) - 60 \cos^{2}(apk)\, ek^{2} \cos(fk) \cos^{3}(wk)$$
$$+ 9 \cos^{2}(apk)\, ek^{2} \cos(5\, fk) \cos^{2}(jk) \cos(wk)$$
$$- 9\, ek^{2} \cos(5\, fk) \cos^{2}(jk) \cos(wk) + 45 \cos^{2}(apk)\, ek \cos(4\, fk) \cos^{2}(jk)$$
$$\cos(wk) - 45\, ek \cos(4\, fk) \cos^{2}(jk) \cos(wk)$$
$$+ 45 \cos^{2}(apk)\, ek^{2} \cos(3\, fk) \cos^{2}(jk) \cos(wk)$$
$$- 45\, ek^{2} \cos(3\, fk) \cos^{2}(jk) \cos(wk) + 60 \cos^{2}(apk) \cos(3\, fk) \cos^{2}(jk) \cos(wk)$$
$$- 60 \cos(3\, fk) \cos^{2}(jk) \cos(wk) + 180 \cos^{2}(apk)\, ek \cos(2\, fk) \cos^{2}(jk) \cos(wk)$$
$$- 180\, ek \cos(2\, fk) \cos^{2}(jk) \cos(wk) + 90 \cos^{2}(apk)\, ek^{2} \cos(fk) \cos^{2}(jk)$$
$$\cos(wk) - 90\, ek^{2} \cos(fk) \cos^{2}(jk) \cos(wk)$$
$$+ 180 \cos^{2}(apk) \cos(fk) \cos^{2}(jk) \cos(wk) - 180 \cos(fk) \cos^{2}(jk) \cos(wk)$$
$$+ 36 \cos(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(4\, fk) \cos(jk) \cos(wk)$$
$$+ 180 \cos(apk) \sin(apk)\, ek \sin(fk) \cos(3\, fk) \cos(jk) \cos(wk)$$
$$+ 176 \cos(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(2\, fk) \cos(jk) \cos(wk)$$
$$+ 240 \cos(apk) \sin(apk) \sin(fk) \cos(2\, fk) \cos(jk) \cos(wk)$$
$$+ 660 \cos(apk) \sin(apk)\, ek \cos(fk) \sin(fk) \cos(jk) \cos(wk)$$
$$+ 268 \cos(apk) \sin(apk)\, ek^{2} \sin(fk) \cos(jk) \cos(wk)$$
$$+ 240 \cos(apk) \sin(apk) \sin(fk) \cos(jk) \cos(wk)$$

$$+ 9 \cos^2(apk)\, ek^2 \cos(5\, fk) \cos(wk) + 45 \cos^2(apk)\, ek \cos(4\, fk) \cos(wk)$$

$$+ 25 \cos^2(apk)\, ek^2 \cos(3\, fk) \cos(wk) + 60 \cos^2(apk) \cos(3\, fk) \cos(wk)$$

$$+ 60 \cos^2(apk)\, ek \cos(2\, fk) \cos(wk) + 30 \cos^2(apk)\, ek^2 \cos(fk) \cos(wk)$$

$$- 60 \cos^2(apk) \cos(fk) \cos(wk))\, zb2/(240\, nk)$$

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Two different procedures for analytically modeling the effects of the Moon's direct gravitational force on artificial Earth satellites are discussed from theoretical and numerical viewpoints. One is developed using classical series expansions of inclination and eccentricity for both the satellite and the Moon, and the other employs the method of averaging.

Both solutions are seen to have advantages, but it is shown that while the former is more accurate in special situations, the latter is quicker and more practical for the general orbit determination problem where observed data are used to correct the orbit in near real time.

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